

# The signed random-to-top operator on tensor space (draft)

Darij Grinberg

August 30, 2016

## 1. Introduction

The purpose of this note is to answer a question I asked in 2010 in [Grinbe10]. It concerns the kernel of a certain operator on the tensor algebra  $T(L)$  of a free module  $L$  over a commutative ring  $\mathbf{k}$  (an operator that picks out a factor from a tensor and moves it to the front, and takes an alternating sum of the results ranging over all factors – an algebraic version of what probabilists call the “random-to-top shuffle”, albeit with signs). Originating in pure curiosity, this question has been tempting me with its apparent connections to the random-to-top and random-to-random shuffling operators as studied in [ReSaWe14] and [Schock02]. I have not (yet?) grown any wiser from these connections, but I was able to answer the question (with some help from a 1950 paper by Specht [Specht50]), and the answer seems (to me) to be interesting enough to warrant some publicity.

We shall **not** use the notations of [Grinbe10] (indeed, our notations in the following will be incompatible with those in [Grinbe10]).

### 1.1. Outline

Let me outline what will be proven in this note. (Everything mentioned here will be defined again in more detail later on.)

We fix a commutative ring  $\mathbf{k}$  and a  $\mathbf{k}$ -module  $L$ , and we consider the tensor algebra  $T(L)$ . We define a  $\mathbf{k}$ -linear map  $\mathbf{t} : T(L) \rightarrow T(L)$  by setting

$$\mathbf{t}(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \sum_{i=1}^k (-1)^{i-1} u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_k$$

for all pure tensors  $u_1 \otimes u_2 \otimes \cdots \otimes u_k \in T(L)$ .<sup>1</sup>

---

<sup>1</sup>This map  $\mathbf{t}$  is the map  $-T$  defined in [Grinbe10].

Roughly speaking, what the map  $\mathbf{t}$  does to a pure tensor can be described as picking out the  $i$ -th tensorand and moving it to the front of the tensor, multiplying the new tensor with  $(-1)^{i-1}$ , and summing the result over all  $i$ 's. Thus, the map  $\mathbf{t}$  is a signed multilinear analogue of the “random-to-top shuffling operator” known from combinatorics (essentially the element  $\sum_{i=1}^k (-1)^{i-1} (1, 2, \dots, i)$  of the group algebra  $\mathbf{k}S_k$ , acting on  $L^{\otimes k}$ ). Alternatively, we can view the restriction of  $\mathbf{t}$  to  $L^{\otimes k}$  as the action of the “random-to-top shuffling element”  $\sum_{i=1}^k (1, 2, \dots, i) \in \mathbf{k}S_k$  (this is the antipode of the  $\Xi_{n,1}$  of [Schock02]) on  $L^{\otimes k}$  via the  $\mathbf{k}S_k$ -module structure on  $L^{\otimes k}$  which is given by permuting the  $k$  tensorands, twisted with the sign representation. For  $L$  a free  $\mathbf{k}$ -module of rank  $\geq k$ , this  $\mathbf{k}S_k$ -module structure is faithful, and so from the behavior of  $\mathbf{t}$  one can draw conclusions about the random-top-shuffling operator.

Our main goal in the first few sections is to describe the kernel of the map  $\mathbf{t}$ . One of our first observations (Proposition 3.3) is that if  $L$  is a free  $\mathbf{k}$ -module, then this kernel is the set of all tensors  $U \in T(L)$  which are annihilated by  $\partial'_g$  for all  $g \in L$ , where the maps  $\partial'_g$  are certain “interior product” operators (see Definition 3.1 for a precise definition). This rather simple fact will come out useful in understanding  $\text{Ker } \mathbf{t}$ .<sup>2</sup>

Once this is proven, we will come to the actual description of  $\text{Ker } \mathbf{t}$ . The tensor algebra  $T(L)$  is  $\mathbb{Z}_2$ -graded, and thus a superalgebra. Thus, any two elements  $U$  and  $V$  of  $T(L)$  have a supercommutator  $[U, V]_s$  (which equals  $UV - (-1)^{nm} VU$  if  $U$  and  $V$  are homogeneous of degrees  $n$  and  $m$ ; otherwise it is determined by  $\mathbf{k}$ -bilinearity). Define

- a sequence  $(L_1, L_2, L_3, \dots)$  of  $\mathbf{k}$ -submodules of  $T(L)$  recursively by  $L_1 = L$  and  $L_{i+1} = [L, L_i]_s$ ;
- a  $\mathbf{k}$ -submodule  $\bar{g}$  of  $T(L)$  by  $\bar{g} = L_2 + L_3 + L_4 + \dots$ ;
- a  $\mathbf{k}$ -submodule  $P$  of  $T(L)$  as the  $\mathbf{k}$ -linear span of all  $xx$  for  $x \in L$ .

(Notice that if 2 is invertible in the ground ring  $\mathbf{k}$ , then  $P \subseteq L_2 \subseteq \bar{g}$ .)

Then,  $\text{Ker } \mathbf{t}$  is the  $\mathbf{k}$ -subalgebra of  $T(L)$  generated by  $\bar{g} + P$ , at least when  $L$  is a free  $\mathbf{k}$ -module. This result (Theorem 6.6 below) will be proven after several auxiliary observations. Our proof will rely on ideas of Wilhelm Specht in his 1950 paper [Specht50] on (what would now be called) PI-algebras (specifically, Sections V and VI of said paper). Specht characterized “properly  $n$ -linear forms”<sup>3</sup>, which, in our notations, would correspond to multilinear elements of  $\text{Ker } \mathbf{t}$  when  $L$  is the free  $\mathbf{k}$ -module  $\mathbf{k}^n$ . (The correspondence is not immediate –

<sup>2</sup>This is very close to what I wrote about bilinear forms in [Grinbe10], but the use of bilinearity instead of linearity was a red herring.

<sup>3</sup>In the original: “eigentlich  $n$ -fach lineare Formen”.

Specht's analogue of the map  $\mathbf{t}$  has no  $(-1)^{i-1}$  signs.) The fact that we consider arbitrary, not just multilinear, elements of  $T(L)$  somewhat complicates our arguments (and prevents us from going as deep as Specht did – e.g., we shall not find a basis for  $\text{Ker } \mathbf{t}$ , although this appears to be doable using Lyndon methods).

Then, we will study an “unsigned” analogue of the map  $\mathbf{t}$ . Namely, we will define a  $\mathbf{k}$ -linear map  $\mathbf{t}' : T(L) \rightarrow T(L)$  by setting

$$\mathbf{t}'(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \sum_{i=1}^k u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_k$$

for all pure tensors  $u_1 \otimes u_2 \otimes \cdots \otimes u_k \in T(L)$ . From a superalgebraic viewpoint,  $\mathbf{t}$  and  $\mathbf{t}'$  are particular cases of a common general construction, but we will witness their kernels behaving differently when the additive group of  $\mathbf{k}$  is not torsionfree. I am not able to describe  $\text{Ker } (\mathbf{t}')$  in the same generality as  $\text{Ker } \mathbf{t}$  (for arbitrary  $\mathbf{k}$ ), but we will see separate descriptions of  $\text{Ker } (\mathbf{t}')$

- in the case when the additive group of  $\mathbf{k}$  is torsionfree (Theorem 7.15), and
- in the case when  $\mathbf{k}$  is a commutative  $\mathbb{F}_p$ -algebra for some prime  $p$  (Theorem 8.10).

Much of our reasoning related to  $\text{Ker } \mathbf{t}$  will apply to  $\text{Ker } (\mathbf{t}')$  as long as some changes are made; supercommutators are replaced by commutators, the  $\mathbf{k}$ -submodule  $P$  is replaced by either 0 (when  $\mathbf{k}$  is torsionfree) or the  $\mathbf{k}$ -submodule of  $T(L)$  spanned by  $x^p$  for all  $x \in L$  (when  $\mathbf{k}$  is an  $\mathbb{F}_p$ -algebra).

It has come to my attention that the description of  $\text{Ker } (\mathbf{t}')$  (Theorem 7.15) in the case when  $\mathbf{k}$  is a  $\mathbb{Q}$ -algebra is a consequence of Amy Pang's [Pang15, Theorem 5.1] (applied to  $\mathcal{H} = T(L)$ ,  $q = 1$  and  $j = 0$ ). (Actually, when  $\mathbf{k}$  is a  $\mathbb{Q}$ -algebra, [Pang15, Theorem 5.1] gives a basis of each eigenspace of  $\mathbf{t}'$ , thus in particular a basis of  $\text{Ker } (\mathbf{t}')$ , but this latter basis is what one would obtain using the symmetrization map and the Poincaré-Birkhoff-Witt theorem from Theorem 7.15. Conversely, Pang's [Pang15, Theorem 5.1] immediately yields Theorem 7.15 when  $\mathbf{k}$  is a  $\mathbb{Q}$ -algebra.)

## 1.2. Acknowledgments

Communications with Franco Saliola (who is studying the random-to-top shuffle as an element of the group algebra of the symmetric group) have helped rekindle my interest in this question. Parts of what comes below might be related (or even equivalent) to some of his recent unpublished work. The SageMath computer algebra system [sage] was used to verify some of the results below (in small degrees and for small ranks of  $L$ ) before a general proof was found.

## 2. The map $\mathbf{t}$

**Convention 2.1.** For the rest of this note, we fix a commutative ring  $\mathbf{k}$ . All unadorned tensor signs (i.e., signs  $\otimes$  without a subscript) in the following are understood to mean  $\otimes_{\mathbf{k}}$ .

We also fix a  $\mathbf{k}$ -module  $L$ .

**Definition 2.2.** Let  $T(L)$  be the tensor algebra of  $L$  (over  $\mathbf{k}$ ). Notice that  $T(L) = L^{\otimes 0} \oplus L^{\otimes 1} \oplus L^{\otimes 2} \oplus \dots$  as  $\mathbf{k}$ -module.

The tensor algebra  $T(L)$  is  $\mathbb{Z}$ -graded (an element of  $L^{\otimes n}$  has degree  $n$ ) and  $\mathbb{Z}_2$ -graded (here an element of  $L^{\otimes n}$  has degree  $n \bmod 2$ ).

(Here, I use  $\mathbb{Z}_2$  to denote the quotient ring  $\mathbb{Z}/2\mathbb{Z}$ , and I use the notation  $n \bmod 2$  to denote the remainder class of  $n$  modulo 2.)

**Definition 2.3.** Let  $\mathbf{t} : T(L) \rightarrow T(L)$  be the  $\mathbf{k}$ -linear map which acts on pure tensors according to the formula

$$\mathbf{t}(u_1 \otimes u_2 \otimes \dots \otimes u_k) = \sum_{i=1}^k (-1)^{i-1} u_i \otimes u_1 \otimes u_2 \otimes \dots \otimes \widehat{u}_i \otimes \dots \otimes u_k$$

(for all  $k \in \mathbb{N}$  and  $u_1, u_2, \dots, u_k \in L$ ), where the  $\widehat{u}_i$  is not an actual tensorand but rather a symbol that means that the factor  $u_i$  is removed from the place where it would usually occur in the tensor product. (This is clearly well-defined.) Thus,  $\mathbf{t}$  is a graded  $\mathbf{k}$ -module endomorphism of  $T(L)$ .

## 3. $\text{Ker } \mathbf{t}$ is the joint kernel of the superderivations

$\partial_g$

**Definition 3.1.** Let  $L^*$  denote the dual  $\mathbf{k}$ -module  $\text{Hom}(L, \mathbf{k})$  of  $L$ . If  $g \in L^*$ , then we define a  $\mathbf{k}$ -linear map  $\partial_g : T(L) \rightarrow T(L)$  by

$$\partial_g(u_1 \otimes u_2 \otimes \dots \otimes u_k) = \sum_{i=1}^k (-1)^{i-1} g(u_i) \cdot u_1 \otimes u_2 \otimes \dots \otimes \widehat{u}_i \otimes \dots \otimes u_k$$

for all  $k \in \mathbb{N}$  and  $u_1, u_2, \dots, u_k \in L$ . (Again, it is easy to check that this is well-defined.)

The map  $\partial_g$  is a lift to  $T(L)$  of what is called the “interior product by  $g$ ” on the Clifford algebra of  $L$  endowed with any quadratic form. This observation provides a motivation for studying  $\partial_g$ ; it will not be used below.

For any  $g \in L^*$ , the map  $\partial_g$  is a superderivation of the superalgebra  $T(L)$ . Rather than explaining these notions, let us state explicitly what the previous sentence means:

**Proposition 3.2.** Let  $g \in L^*$ .

(a) Then,  $\partial_g(1) = 0$ .

(b) Also, if  $n \in \mathbb{N}$ ,  $a \in L^{\otimes n}$  and  $b \in T(L)$ , then  $\partial_g(ab) = \partial_g(a)b + (-1)^n a\partial_g(b)$ .

*Proof of Proposition 3.2.* We give this straightforward proof purely for the sake of completeness.

(a) The unity 1 of the ring  $T(L)$  is the empty tensor product. The definition of  $\partial_g$  thus shows that  $\partial_g(1)$  is an empty sum, and therefore equal to 0. This proves Proposition 3.2 (a).

(b) Let  $n \in \mathbb{N}$ ,  $a \in L^{\otimes n}$  and  $b \in T(L)$ . We need to prove the equality  $\partial_g(ab) = \partial_g(a)b + (-1)^n a\partial_g(b)$ . Since this equality is  $\mathbf{k}$ -linear in each of  $a$  and  $b$ , we can WLOG assume that both  $a$  and  $b$  are pure tensors. Assume this. Since  $a \in L^{\otimes n}$  is a pure tensor, we have  $a = a_1 \otimes a_2 \otimes \cdots \otimes a_n$  for some  $a_1, a_2, \dots, a_n \in L$ . Consider these  $a_1, a_2, \dots, a_n$ . Since  $b$  is a pure tensor, we have  $b = b_1 \otimes b_2 \otimes \cdots \otimes b_m$  for some  $m \in \mathbb{N}$  and  $b_1, b_2, \dots, b_m \in L$ . Consider this  $m$  and these  $b_1, b_2, \dots, b_m$ . Multiplying the equalities  $a = a_1 \otimes a_2 \otimes \cdots \otimes a_n$  and  $b = b_1 \otimes b_2 \otimes \cdots \otimes b_m$ , we obtain

$$\begin{aligned} ab &= (a_1 \otimes a_2 \otimes \cdots \otimes a_n) (b_1 \otimes b_2 \otimes \cdots \otimes b_m) \\ &= a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_m. \end{aligned}$$

Hence,

$$\begin{aligned}
& \partial_g(ab) \\
&= \partial_g(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_m) \\
&= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot \underbrace{a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_m}_{=(a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n) \cdot (b_1 \otimes b_2 \otimes \cdots \otimes b_m)} \\
&\quad + \sum_{i=n+1}^{n+m} (-1)^{i-1} g(b_{i-n}) \cdot \underbrace{a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_{i-n}} \otimes \cdots \otimes b_m}_{=(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_{i-n}} \otimes \cdots \otimes b_m)} \\
&\quad \text{(by the definition of } \partial_g) \\
&= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n) \cdot \underbrace{(b_1 \otimes b_2 \otimes \cdots \otimes b_m)}_{=b} \\
&\quad + \sum_{i=n+1}^{n+m} (-1)^{i-1} g(b_{i-n}) \cdot \underbrace{(a_1 \otimes a_2 \otimes \cdots \otimes a_n)}_{=a} \cdot (b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_{i-n}} \otimes \cdots \otimes b_m) \\
&= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n) \cdot b \\
&\quad + \sum_{i=n+1}^{n+m} (-1)^{i-1} g(b_{i-n}) \cdot a \cdot (b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_{i-n}} \otimes \cdots \otimes b_m) \\
&= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n) \cdot b \\
&\quad + \sum_{i=1}^m (-1)^{i+n-1} g(b_i) \cdot a \cdot (b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_i} \otimes \cdots \otimes b_m) \tag{1} \\
&\quad \text{(here, we have substituted } i \text{ for } i-n \text{ in the second sum).}
\end{aligned}$$

□

But  $a = a_1 \otimes a_2 \otimes \cdots \otimes a_n$  shows that

$$\begin{aligned}
\partial_g(a) &= \partial_g(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \\
&= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n
\end{aligned}$$

(by the definition of  $\partial_g$ ). Similarly,

$$\partial_g(b) = \sum_{i=1}^m (-1)^{i-1} g(b_i) \cdot b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_i} \otimes \cdots \otimes b_m.$$

Hence,

$$\begin{aligned}
& \underbrace{\partial_g(a)}_{= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n} \quad b + (-1)^n a \quad \underbrace{\partial_g(b)}_{= \sum_{i=1}^m (-1)^{i-1} g(b_i) \cdot b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_i} \otimes \cdots \otimes b_m} \\
&= \underbrace{\left( \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n \right) b}_{= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n) \cdot b} \\
&\quad + \underbrace{(-1)^n a \sum_{i=1}^m (-1)^{i-1} g(b_i) \cdot b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_i} \otimes \cdots \otimes b_m}_{= \sum_{i=1}^m (-1)^{i+n-1} g(b_i) \cdot a \cdot (b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_i} \otimes \cdots \otimes b_m)} \\
&= \sum_{i=1}^n (-1)^{i-1} g(a_i) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_n) \cdot b \\
&\quad + \sum_{i=1}^m (-1)^{i+n-1} g(b_i) \cdot a \cdot (b_1 \otimes b_2 \otimes \cdots \otimes \widehat{b_i} \otimes \cdots \otimes b_m) \\
&= \partial_g(ab) \quad (\text{by (1)}).
\end{aligned}$$

This proves Proposition 3.2 (b).

The maps  $\partial_g$  relate to  $\text{Ker } \mathbf{t}$  as follows:

**Proposition 3.3. (a)** We have  $\partial_g(\text{Ker } \mathbf{t}) = 0$  for every  $g \in L^*$ .

**(b)** Assume that  $L$  is a free  $\mathbf{k}$ -module. Then,

$$\text{Ker } \mathbf{t} = \{U \in T(L) \mid \partial_g(U) = 0 \text{ for every } g \in L^*\}.$$

*Proof of Proposition 3.3.* For every  $g \in L^*$ , we define a  $\mathbf{k}$ -module homomorphism  $\mathbf{c}_g : T(L) \rightarrow T(L)$  by the formula

$$\mathbf{c}_g(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \begin{cases} 0, & \text{if } k = 0; \\ g(u_1) u_2 \otimes u_3 \otimes \cdots \otimes u_k, & \text{if } k > 0 \end{cases}$$

for all  $k \in \mathbb{N}$  and  $u_1, u_2, \dots, u_k \in L$ . (Again, this is well-defined for rather obvious reasons.)

It is now easy to prove that  $\partial_g = \mathbf{c}_g \circ \mathbf{t}$  for every  $g \in L^*$ <sup>4</sup>. Thus, every  $g \in L^*$  satisfies  $\text{Ker } \mathbf{t} \subseteq \text{Ker } (\partial_g)$ , so that  $\partial_g(\text{Ker } \mathbf{t}) = 0$ . This proves Proposition 3.3 (a).

**(b)** The definition of  $\mathbf{c}_g$  easily yields

$$\mathbf{c}_g(vU) = g(v) U \quad \text{for any } v \in L \text{ and } U \in T(L). \quad (2)$$

<sup>4</sup>Indeed,  $\partial_g$  and  $\mathbf{c}_g \circ \mathbf{t}$  are two  $\mathbf{k}$ -linear maps which equal each other on each pure tensor (this can be checked readily).

We denote by  $\overline{T(L)}$  the  $\mathbf{k}$ -submodule  $L^{\otimes 1} \oplus L^{\otimes 2} \oplus L^{\otimes 3} \oplus \dots$  of  $T(L)$ . We notice that  $\mathbf{t}(T(L)) \subseteq \overline{T(L)}$  (since  $\mathbf{t}$  is a graded map which equals 0 in degree 0).

Now, let  $V \in \{U \in T(L) \mid \partial_g(U) = 0 \text{ for every } g \in L^*\}$ . We are going to show that  $V \in \text{Ker } \mathbf{t}$ .

We have  $V \in \{U \in T(L) \mid \partial_g(U) = 0 \text{ for every } g \in L^*\}$ . Hence,  $V \in T(L)$ , and we have  $\partial_g(V) = 0$  for every  $g \in L^*$ .

We fix a basis  $(e_i)_{i \in I}$  of the  $\mathbf{k}$ -module  $L$  (this exists since  $L$  is free). For every  $i \in I$ , let  $e_i^* \in L^*$  be the  $\mathbf{k}$ -linear map  $L \rightarrow \mathbf{k}$  which sends  $e_i$  to 1 and sends all other  $e_j$  to 0. In other words,  $e_i^* \in L^*$  satisfies  $e_i^*(e_j) = \delta_{j,i}$  for all  $j \in I$ . (If  $I$  is finite, then  $(e_i^*)_{i \in I}$  is thus the basis of  $L^*$  dual to the basis  $(e_i)_{i \in I}$  of  $L$ . For arbitrary  $I$ , it might not be a basis of  $L^*$ , but we don't care.)

We have  $\mathbf{t}(V) \in \mathbf{t}(T(L)) \subseteq \overline{T(L)}$ . Thus, we can write the tensor  $\mathbf{t}(V)$  in the form  $\mathbf{t}(V) = \sum_{i \in I} e_i V_i$  for some tensors  $V_i \in T(L)$  (all but finitely many of which are zero). Consider these  $V_i$ . For every  $j \in I$ , we can apply the map  $\mathbf{c}_{e_j^*}$  to both sides of the equality  $\mathbf{t}(V) = \sum_{i \in I} e_i V_i$  and obtain

$$\mathbf{c}_{e_j^*}(\mathbf{t}(V)) = \mathbf{c}_{e_j^*}\left(\sum_{i \in I} e_i V_i\right) = \sum_{i \in I} \underbrace{\mathbf{c}_{e_j^*}(e_i V_i)}_{\substack{= e_j^*(e_i) V_i \\ \text{(by (2))}}} = \sum_{i \in I} \underbrace{e_j^*(e_i)}_{=\delta_{i,j}} V_i = \sum_{i \in I} \delta_{i,j} V_i = V_j.$$

But every  $g \in L^*$  satisfies  $\partial_g(V) = 0$ . Since  $\underbrace{\partial_g}_{=\mathbf{c}_g \circ \mathbf{t}}(V) = \mathbf{c}_g(\mathbf{t}(V))$ , this rewrites

as follows: Every  $g \in L^*$  satisfies  $\mathbf{c}_g(\mathbf{t}(V)) = 0$ . Applied to  $g = e_j^*$ , this yields  $\mathbf{c}_{e_j^*}(\mathbf{t}(V)) = 0$  for every  $j \in I$ . Compared with  $\mathbf{c}_{e_j^*}(\mathbf{t}(V)) = V_j$ , this yields  $V_j = 0$ . This holds for each  $j \in I$ . Thus,  $V_i = 0$  for each  $i \in I$ . Thus,  $\mathbf{t}(V) = \sum_{i \in I} e_i \underbrace{V_i}_{=0} = 0$ , so that  $V \in \text{Ker } \mathbf{t}$ .

Now, let us forget that we fixed  $V$ . We thus have shown that  $V \in \text{Ker } \mathbf{t}$  for every  $V \in \{U \in T(L) \mid \partial_g(U) = 0 \text{ for every } g \in L^*\}$ . In other words, we have shown the inclusion

$$\{U \in T(L) \mid \partial_g(U) = 0 \text{ for every } g \in L^*\} \subseteq \text{Ker } \mathbf{t}.$$

The reverse inclusion also holds (its proof is a trivial application of  $\partial_g = \mathbf{c}_g \circ \mathbf{t}$ ). Combined, the two inclusions yield

$$\text{Ker } \mathbf{t} = \{U \in T(L) \mid \partial_g(U) = 0 \text{ for every } g \in L^*\}.$$

Proposition 3.3 (b) is thus proven.  $\square$



## 4. $\text{Ker } \mathbf{t}$ is a subalgebra of $T(L)$

We now want to describe  $\text{Ker } \mathbf{t}$ . Clearly,  $\text{Ker } \mathbf{t}$  is a graded  $\mathbf{k}$ -submodule of  $T(L)$  (since  $\mathbf{t}$  is a graded map). We first introduce some notations:

**Definition 4.1.** We define a  $\mathbf{k}$ -bilinear map  $\text{scomm} : T(L) \times T(L) \rightarrow T(L)$  as follows: We set

$$\text{scomm}(U, V) = UV - (-1)^{nm} VU$$

for any  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $U \in L^{\otimes n}$  and  $V \in L^{\otimes m}$ . (This is easily seen to be well-defined.)

If  $U \in T(L)$  and  $V \in T(L)$ , then we denote the tensor  $\text{scomm}(U, V)$  by  $[U, V]_s$ , and we call it the *supercommutator* of  $U$  and  $V$ . Thus,  $[U, V]_s$  depends  $\mathbf{k}$ -linearly on each of  $U$  and  $V$ , and satisfies

$$[U, V]_s = UV - (-1)^{nm} VU$$

for any  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $U \in L^{\otimes n}$  and  $V \in L^{\otimes m}$ .

This definition should not surprise anyone familiar with superalgebras. In fact, recall that the  $\mathbf{k}$ -algebra  $T(L)$  is  $\mathbb{Z}_2$ -graded; thus,  $T(L)$  is a  $\mathbf{k}$ -superalgebra. Consequently, it has a supercommutator. This supercommutator is precisely the map  $\text{scomm}$  that we have just defined. We just preferred not to use the language of superalgebras.

Clearly,  $[U, V]_s = -(-1)^{nm} [V, U]_s$  for any  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $U \in L^{\otimes n}$  and  $V \in L^{\otimes m}$ . The supercommutator  $\text{scomm}$  (or, differently written,  $[\cdot, \cdot]_s$ ) furthermore satisfies the following analogue of the Leibniz and Jacobi identities:

**Proposition 4.2.** Let  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $U \in L^{\otimes n}$ ,  $V \in L^{\otimes m}$  and  $W \in T(L)$ . Then:

- (a) We have  $[U, VW]_s = [U, V]_s W + (-1)^{nm} V [U, W]_s$ .
- (b) We have  $[U, [V, W]_s]_s = [[U, V]_s, W]_s + (-1)^{nm} [V, [U, W]_s]_s$ .

*Proof of Proposition 4.2.* This is straightforward and left to the reader.  $\square$

**Definition 4.3.** We will apply the same notations to supercommutators that are usually applied to commutators. For instance, when  $P$  and  $Q$  are two  $\mathbf{k}$ -submodules of  $T(L)$ , we will use the notation  $[P, Q]_s$  for the  $\mathbf{k}$ -linear span of the supercommutators  $[U, V]_s$  for  $U \in P$  and  $V \in Q$  (just as one commonly writes  $[P, Q]$  for the  $\mathbf{k}$ -linear span of the commutators  $[U, V]$  for  $U \in P$  and  $V \in Q$ ).

**Proposition 4.4.** (a) We have  $L^{\otimes 0} \subseteq \text{Ker } \mathbf{t}$ .

- (b) We have  $\text{Ker } \mathbf{t} \cdot \text{Ker } \mathbf{t} \subseteq \text{Ker } \mathbf{t}$ .
- (c) We have  $[L, L]_s \subseteq \text{Ker } \mathbf{t}$ .
- (d) We have  $[L, \text{Ker } \mathbf{t}]_s \subseteq \text{Ker } \mathbf{t}$ .
- (e) We have  $xx \in \text{Ker } \mathbf{t}$  for each  $x \in L$ .

*Proof of Proposition 4.4. (a)* This is obvious.

**(b)** For every  $U \in T(L)$ , we define a  $\mathbf{k}$ -linear map  $\mathbf{i}_U : T(L) \rightarrow T(L)$  by the formula

$$\mathbf{i}_U(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \begin{cases} 0, & \text{if } k = 0; \\ v_1 \cdot U \cdot (v_2 \otimes v_3 \otimes \cdots \otimes v_k), & \text{if } k > 0 \end{cases}$$

for all  $k \in \mathbb{N}$  and  $v_1, v_2, \dots, v_k \in L$ . Then, it is straightforward to see that

$$\mathbf{t}(UV) = \mathbf{t}(U)V + (-1)^n \mathbf{i}_U(\mathbf{t}(V)) \quad (3)$$

for all  $n \in \mathbb{N}$ ,  $U \in L^{\otimes n}$  and  $V \in T(L)$  <sup>5</sup>.

---

<sup>5</sup>*Proof of (3):* Let  $n \in \mathbb{N}$ ,  $U \in L^{\otimes n}$  and  $V \in T(L)$ . We need to prove the equality (3). Since this equality is  $\mathbf{k}$ -linear in each of  $U$  and  $V$  (because  $\mathbf{i}_U$  is  $\mathbf{k}$ -linear in  $U$ ), we can WLOG assume that both  $U$  and  $V$  are pure tensors. Assume this. Since  $U \in L^{\otimes n}$  is a pure tensor, we have  $U = u_1 \otimes u_2 \otimes \cdots \otimes u_n$  for some  $u_1, u_2, \dots, u_n \in L$ . Consider these  $u_1, u_2, \dots, u_n$ . Since  $V$  is a pure tensor, we have  $V = v_1 \otimes v_2 \otimes \cdots \otimes v_m$  for some  $m \in \mathbb{N}$  and  $v_1, v_2, \dots, v_m \in L$ . Consider this  $m$  and these  $v_1, v_2, \dots, v_m$ . Multiplying the equalities  $U = u_1 \otimes u_2 \otimes \cdots \otimes u_n$  and  $V = v_1 \otimes v_2 \otimes \cdots \otimes v_m$ , we obtain

$$\begin{aligned} UV &= (u_1 \otimes u_2 \otimes \cdots \otimes u_n)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) \\ &= u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_m. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{t}(UV) &= \mathbf{t}(u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_m) \\ &= \sum_{i=1}^n (-1)^{i-1} \underbrace{u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_m}_{=(u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_m)} \\ &\quad + \sum_{i=n+1}^{n+m} (-1)^{i-1} \underbrace{v_{i-n} \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v_{i-n}} \otimes \cdots \otimes v_m}_{=v_{i-n} \cdot (u_1 \otimes u_2 \otimes \cdots \otimes u_n) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v_{i-n}} \otimes \cdots \otimes v_m)} \\ &\quad \text{(by the definition of } \mathbf{t} \text{)} \\ &= \sum_{i=1}^n (-1)^{i-1} (u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n) \cdot \underbrace{(v_1 \otimes v_2 \otimes \cdots \otimes v_m)}_{=V} \\ &\quad + \sum_{i=n+1}^{n+m} (-1)^{i-1} v_{i-n} \cdot \underbrace{(u_1 \otimes u_2 \otimes \cdots \otimes u_n)}_{=U} \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v_{i-n}} \otimes \cdots \otimes v_m) \\ &= \sum_{i=1}^n (-1)^{i-1} (u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n) \cdot V \\ &\quad + \sum_{i=n+1}^{n+m} (-1)^{i-1} v_{i-n} \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v_{i-n}} \otimes \cdots \otimes v_m) \\ &= \sum_{i=1}^n (-1)^{i-1} (u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n) \cdot V \\ &\quad + \sum_{i=1}^m (-1)^{i+n-1} v_i \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_m) \quad (4) \\ &\quad \text{(here, we have substituted } i \text{ for } i-n \text{ in the second sum).} \end{aligned}$$

Now, we need to prove  $\text{Ker } \mathbf{t} \cdot \text{Ker } \mathbf{t} \subseteq \text{Ker } \mathbf{t}$ . In other words, we need to prove that  $UV \in \text{Ker } \mathbf{t}$  for all  $U \in \text{Ker } \mathbf{t}$  and  $V \in \text{Ker } \mathbf{t}$ . So let us fix  $U \in \text{Ker } \mathbf{t}$  and  $V \in \text{Ker } \mathbf{t}$ . Since  $\text{Ker } \mathbf{t}$  is a graded  $\mathbf{k}$ -submodule of  $T(L)$  (because  $\mathbf{t}$  is a graded map), we can WLOG assume that  $U$  is homogeneous, i.e., that  $U \in L^{\otimes n}$  for some  $n \in \mathbb{N}$ . Assume this, and consider this  $n$ . From (3), we thus obtain

---

But  $U = u_1 \otimes u_2 \otimes \cdots \otimes u_n$  shows that

$$\begin{aligned} \mathbf{t}(U) &= \mathbf{t}(u_1 \otimes u_2 \otimes \cdots \otimes u_n) \\ &= \sum_{i=1}^n (-1)^{i-1} u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u}_i \otimes \cdots \otimes u_n \end{aligned}$$

(by the definition of  $\mathbf{t}$ ). Similarly,

$$\mathbf{t}(V) = \sum_{i=1}^m (-1)^{i-1} v_i \otimes v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m.$$

Applying the map  $\mathbf{i}_U$  to both sides of this equality, we obtain

$$\begin{aligned} \mathbf{i}_U(\mathbf{t}(V)) &= \mathbf{i}_U \left( \sum_{i=1}^m (-1)^{i-1} v_i \otimes v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m \right) \\ &= \sum_{i=1}^m (-1)^{i-1} \underbrace{\mathbf{i}_U(v_i \otimes v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m)}_{=v_i \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m)} \\ &\quad \text{(by the definition of } \mathbf{i}_U, \text{ since } m > 0 \text{ (because } i \in \{1, 2, \dots, m\} \text{))} \\ &= \sum_{i=1}^m (-1)^{i-1} v_i \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m). \end{aligned}$$

Hence,

$$\begin{aligned} &\underbrace{\mathbf{t}(U)}_{= \sum_{i=1}^n (-1)^{i-1} u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u}_i \otimes \cdots \otimes u_n} \quad V + (-1)^n \quad \underbrace{\mathbf{i}_U(\mathbf{t}(V))}_{= \sum_{i=1}^m (-1)^{i-1} v_i \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m)} \\ &= \left( \sum_{i=1}^n (-1)^{i-1} u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u}_i \otimes \cdots \otimes u_n \right) V \\ &\quad = \sum_{i=1}^n (-1)^{i-1} (u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u}_i \otimes \cdots \otimes u_n) \cdot V \\ &\quad + (-1)^n \underbrace{\sum_{i=1}^m (-1)^{i-1} v_i \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m)}_{= \sum_{i=1}^m (-1)^{i+n-1} v_i \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m)} \\ &= \sum_{i=1}^n (-1)^{i-1} (u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u}_i \otimes \cdots \otimes u_n) \cdot V \\ &\quad + \sum_{i=1}^m (-1)^{i+n-1} v_i \cdot U \cdot (v_1 \otimes v_2 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m) \\ &= \mathbf{t}(UV) \quad \text{(by (4))}. \end{aligned}$$

This proves (3).

$$\mathbf{t}(UV) = \underbrace{\mathbf{t}(U)}_{\substack{=0 \\ (\text{since } U \in \text{Ker } \mathbf{t})}} V + (-1)^n \mathbf{i}_U \left( \underbrace{\mathbf{t}(V)}_{\substack{=0 \\ (\text{since } V \in \text{Ker } \mathbf{t})}} \right) = 0, \text{ so that } UV \in \text{Ker } \mathbf{t}, \text{ just}$$

as we wished to prove. Proposition 4.4 (b) is thus proven.

(c) This is straightforward: For all  $x, y \in L$ , we have  $[x, y]_s = xy - (-1)^{1 \cdot 1} yx = xy + yx$  and  $\mathbf{t}(xy) = xy - yx$  and  $\mathbf{t}(yx) = yx - xy$ . Thus, for all  $x, y \in L$ , we have

$$\mathbf{t} \left( \underbrace{[x, y]_s}_{=xy+yx} \right) = \underbrace{\mathbf{t}(xy)}_{=xy-yx} + \underbrace{\mathbf{t}(yx)}_{=yx-xy} = (xy - yx) + (yx - xy) = 0,$$

and thus  $[x, y]_s \in \text{Ker } \mathbf{t}$ . In other words,  $[L, L]_s \subseteq \text{Ker } \mathbf{t}$ .

(d) It is enough to show that  $[u, V]_s \in \text{Ker } \mathbf{t}$  for every  $u \in L$  and  $V \in \text{Ker } \mathbf{t}$ . So let  $u \in L$  and  $V \in \text{Ker } \mathbf{t}$ . Then,  $\mathbf{t}(V) = 0$ .

Since  $\text{Ker } \mathbf{t}$  is a graded  $\mathbf{k}$ -submodule of  $T(L)$ , we WLOG assume that  $V$  is homogeneous. That is,  $V \in L^{\otimes m}$  for some  $m \in \mathbb{N}$ . Consider this  $m$ . Applying (3) to  $n = 1$  and  $U = u$ , we obtain

$$\mathbf{t}(uV) = \underbrace{\mathbf{t}(u)}_{=u} V + (-1)^1 \mathbf{i}_u \left( \underbrace{\mathbf{t}(V)}_{=0} \right) = uV.$$

On the other hand, we can apply (3) to  $m$ ,  $V$  and  $u$  instead of  $n$ ,  $U$  and  $V$ . As a result, we obtain

$$\mathbf{t}(Vu) = \underbrace{\mathbf{t}(V)}_{=0} u + (-1)^m \mathbf{i}_V \left( \underbrace{\mathbf{t}(u)}_{=u} \right) = (-1)^m \underbrace{\mathbf{i}_V(u)}_{\substack{=uV \\ (\text{by the definition of } \mathbf{i}_V)}} = (-1)^m uV.$$

Now,

$$\mathbf{t} \left( \underbrace{[u, V]_s}_{=uV - (-1)^{1 \cdot m} Vu} \right) = \underbrace{\mathbf{t}(uV)}_{=uV} - (-1)^{1 \cdot m} \underbrace{\mathbf{t}(Vu)}_{=(-1)^m uV} = uV - \underbrace{(-1)^{1 \cdot m} (-1)^m}_{=1} uV = 0,$$

so that  $[u, V]_s \in \text{Ker } \mathbf{t}$ . This completes our proof of Proposition 4.4 (d).

(e) This is also straightforward. □

## 5. The submodules $\bar{\mathfrak{g}}$ , $P$ and $\mathfrak{h}$

Parts (a) and (b) of Proposition 4.4 show that  $\text{Ker } \mathbf{t}$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ . Parts (c) and (d) show that nontrivial iterated supercommutators of elements of

$L$  (that is, tensors in  $[L, L]_s$  or  $[L, [L, L]_s]_s$ , etc.) belong to  $\text{Ker } \mathbf{t}$ , and (by parts (a) and (b)) so do their products. Part (e) shows that elements of the form  $xx$  with  $x \in L$  are in  $\text{Ker } \mathbf{t}$  as well. Of course,  $\mathbf{k}$ -linear combinations of elements of  $\text{Ker } \mathbf{t}$  are also elements of  $\text{Ker } \mathbf{t}$ . Our goal is to show that all elements of  $\text{Ker } \mathbf{t}$  are obtained in these ways. We shall, however, first formalize and somewhat improve this goal.

**Definition 5.1.** We recursively define a sequence  $(L_1, L_2, L_3, \dots)$  of  $\mathbf{k}$ -submodules of  $T(L)$  as follows: We set  $L_1 = L$ , and  $L_{i+1} = [L, L_i]_s$  for every positive integer  $i$ .

For instance,  $L_2 = [L, L]_s$  and  $L_3 = [L, L_2]_s = [L, [L, L]_s]_s$ .

If you are familiar with Lie superalgebras, you will recognize  $L_1 + L_2 + L_3 + \dots$  as the Lie subsuperalgebra of  $T(L)$  generated by  $L$ . I suspect that it is the free Lie superalgebra over  $L$  (though I am not sure if this is unconditionally true).

By induction, it is clear that  $L_i \subseteq L^{\otimes i}$  for every positive integer  $i$ .

**Definition 5.2.** Let  $\bar{\mathfrak{g}}$  denote the  $\mathbf{k}$ -submodule  $L_2 + L_3 + L_4 + \dots$  of  $T(L)$ .

It is easy to see (but unnecessary for us) that  $\bar{\mathfrak{g}}$  is a Lie superalgebra under the supercommutator  $[\cdot, \cdot]_s$ . (See Proposition 5.7 below.)

**Definition 5.3.** Let  $P$  denote the  $\mathbf{k}$ -submodule of  $L^{\otimes 2}$  spanned by elements of the form  $x \otimes x$  with  $x \in L$ . Notice that  $x \otimes x = xx$  in the  $\mathbf{k}$ -algebra  $T(L)$  for every  $x \in L$ .

**Proposition 5.4.** If 2 is invertible in  $\mathbf{k}$ , then  $P \subseteq L_2 \subseteq \bar{\mathfrak{g}}$ .

*Proof of Proposition 5.4.* Assume that 2 is invertible in  $\mathbf{k}$ . Let  $x \in L$ . Then,  $xx = x \otimes x$  in  $T(L)$ , and since  $x$  has degree 1, we have  $[x, x]_s = xx - (-1)^{1 \cdot 1} xx =$

$$xx + xx = 2xx = 2x \otimes x. \text{ Hence, } x \otimes x = \frac{1}{2} \left[ \underbrace{x}_{\in L}, \underbrace{x}_{\in L} \right]_s \in \frac{1}{2} \underbrace{[L, L]_s}_{=L_2} \subseteq L_2. \text{ Since}$$

we have proven this for each  $x \in L$ , we thus obtain  $P \subseteq L_2$  (since  $P$  is spanned by the  $x \otimes x$  with  $x \in L$ ). Combined with  $L_2 \subseteq \bar{\mathfrak{g}}$ , this proves Proposition 5.4.  $\square$

**Definition 5.5.** Let  $\mathfrak{h} = \bar{\mathfrak{g}} + P$ .

Proposition 5.4 shows that  $\mathfrak{h} = \bar{\mathfrak{g}}$  if 2 is invertible in  $\mathbf{k}$ . (But in general,  $\mathfrak{h}$  can be larger than  $\bar{\mathfrak{g}}$ .) Obviously,  $\bar{\mathfrak{g}} \subseteq \mathfrak{h}$  and  $P \subseteq \mathfrak{h}$ .

**Proposition 5.6. (a)** We have  $[L, \bar{\mathfrak{g}}]_s \subseteq \bar{\mathfrak{g}}$ .

**(b)** Furthermore,  $[L, P]_s \subseteq \bar{\mathfrak{g}}$  and  $[L, \mathfrak{h}]_s \subseteq \bar{\mathfrak{g}} \subseteq \mathfrak{h}$ .

*Proof of Proposition 5.6. (a)* Since  $\bar{\mathfrak{g}} = L_2 + L_3 + L_4 + \cdots = \sum_{i \geq 2} L_i$ , we have

$$[L, \bar{\mathfrak{g}}]_s = \left[ L, \sum_{i \geq 2} L_i \right]_s = \sum_{i \geq 2} \underbrace{[L, L_i]_s}_{=L_{i+1}} = \sum_{i \geq 2} L_{i+1} = \sum_{i \geq 3} L_i \subseteq \sum_{i \geq 2} L_i = \bar{\mathfrak{g}}.$$

Thus, Proposition 5.6 (a) is proven.

Also, we have  $[\bar{\mathfrak{g}}, L]_s = [L, \bar{\mathfrak{g}}]_s \subseteq \bar{\mathfrak{g}}$ .

(b) It is easy to check that

$$[U, xx]_s = [[U, x]_s, x]_s \quad (5)$$

for every  $U \in T(L)$  and every  $x \in L$ . Hence, for every  $U \in L$  and  $x \in L$ , we have

$$[U, xx]_s = \left[ \left[ \underbrace{U}_{\in L}, \underbrace{x}_{\in L} \right]_s, \underbrace{x}_{\in L} \right]_s \in \left[ \underbrace{[L, L]_s}_{=L_2 \subseteq \bar{\mathfrak{g}}}, L \right]_s \subseteq [\bar{\mathfrak{g}}, L]_s \subseteq \bar{\mathfrak{g}}.$$

Thus,  $[L, P]_s \subseteq \bar{\mathfrak{g}}$  (since  $P$  is the  $\mathbf{k}$ -linear span of all elements of  $T(L)$  of the form  $x \otimes x = xx$  with  $x \in L$ ).

Since  $\mathfrak{h} = \bar{\mathfrak{g}} + P$ , we have  $[L, \mathfrak{h}]_s = \underbrace{[L, \bar{\mathfrak{g}}]_s}_{\subseteq \bar{\mathfrak{g}}} + \underbrace{[L, P]_s}_{\subseteq \bar{\mathfrak{g}}} \subseteq \bar{\mathfrak{g}} + \bar{\mathfrak{g}} \subseteq \bar{\mathfrak{g}} \subseteq \mathfrak{h}$ . This

finishes the proof of Proposition 5.6 (b).  $\square$

The following proposition will not be used in the following, but provides an interesting aside (and explains why we are using Fraktur letters for  $\bar{\mathfrak{g}}$  and  $\mathfrak{h}$ ):

**Proposition 5.7. (a)** We have  $[\mathfrak{h}, \mathfrak{h}]_s \subseteq \bar{\mathfrak{g}} \subseteq \mathfrak{h}$ .

(b) The four  $\mathbf{k}$ -submodules  $\bar{\mathfrak{g}}$ ,  $\mathfrak{h}$ ,  $\bar{\mathfrak{g}} + L$  and  $\mathfrak{h} + L$  of  $T(L)$  are invariant under the supercommutator  $[\cdot, \cdot]_s$ . (In superalgebraic terms, they are Lie subsuperalgebras of  $T(L)$  (with the supercommutator  $[\cdot, \cdot]_s$  as the Lie bracket).)

*Proof of Proposition 5.7. (a)* First, we notice that

$$\left[ P, \sum_{i \geq 1} L_i \right]_s \subseteq \bar{\mathfrak{g}}. \quad (6)$$

6

Next, we notice that any two positive integers  $i$  and  $j$  satisfy

$$[L_i, L_j]_s \subseteq L_{i+j}. \quad (7)$$

---

<sup>6</sup>*Proof of (6):* It is clearly enough to show that  $[P, L_i]_s \subseteq \bar{\mathfrak{g}}$  for all positive integers  $i$ . So let us do this. Let  $i$  be a positive integer. We need to show that  $[P, L_i]_s \subseteq \bar{\mathfrak{g}}$ . In other words, we need to show that  $[xx, L_i]_s \subseteq \bar{\mathfrak{g}}$  for every  $x \in L$  (because the  $\mathbf{k}$ -module  $P$  is spanned by elements of

---

the form  $x \otimes x = xx$  with  $x \in L$ ). So fix  $x \in L$ . Then,

$$\begin{aligned}
[xx, L_i]_s &= [L_i, xx]_s = \left[ \left[ L_i, \underbrace{x}_{\in L} \right]_s, \underbrace{x}_{\in L} \right]_s && \text{(by (5))} \\
&\subseteq \left[ \underbrace{[L_i, L]_s}_{=[L, L_i]_s = L_{i+1}}, L \right]_s = [L_{i+1}, L]_s = [L, L_{i+1}]_s = L_{i+2} \\
&\subseteq L_2 + L_3 + L_4 + \cdots = \bar{\mathfrak{g}}.
\end{aligned}$$

This completes our proof of (6).

<sup>7</sup> Now,

$$\begin{aligned}
\left[ \sum_{i \geq 1} L_i, \sum_{i \geq 1} L_i \right]_s &= \left[ \sum_{i \geq 1} L_i, \sum_{j \geq 1} L_j \right]_s = \sum_{i \geq 1} \sum_{j \geq 1} \underbrace{[L_i, L_j]_s}_{\substack{\subseteq L_{i+j} \\ \text{(by (7))}}} \subseteq \sum_{i \geq 1} \sum_{j \geq 1} L_{i+j} \\
&\subseteq \sum_{k \geq 2} L_k \quad (\text{since } i+j \geq 2 \text{ for any } i \geq 1 \text{ and } j \geq 1) \\
&= L_2 + L_3 + L_4 + \cdots = \bar{\mathfrak{g}}.
\end{aligned}$$

Recall now that  $\bar{\mathfrak{g}} = L_2 + L_3 + L_4 + \cdots = \sum_{i \geq 2} L_i \subseteq \sum_{i \geq 1} L_i$ . Thus,

$$\left[ \bar{\mathfrak{g}}, \sum_{i \geq 1} L_i \right]_s \subseteq \left[ \sum_{i \geq 1} L_i, \sum_{i \geq 1} L_i \right]_s \subseteq \bar{\mathfrak{g}}.$$

---

<sup>7</sup>Proof of (7): We shall prove (7) by induction over  $j$ :

*Induction base:* For every positive integer  $i$ , we have  $\left[ L_i, \underbrace{L_1}_{=L} \right]_s = [L_i, L]_s = [L, L_i]_s = L_{i+1}$ .

Thus, (7) holds for  $j = 1$ . This completes the induction base.

*Induction step:* Let  $J$  be a positive integer. Assume that (7) is proven for  $j = J$ . We now need to prove (7) for  $j = J + 1$ .

We have assumed that (7) is proven for  $j = J$ . In other words,

$$[L_i, L_J]_s \subseteq L_{i+J} \quad \text{for every positive integer } i. \quad (8)$$

Now, let  $i$  be a positive integer. Then,  $L_i$ ,  $L_J$  and  $L$  are graded  $\mathbf{k}$ -submodules of  $T(L)$ , and we have

$$\begin{aligned}
\left[ L_i, \underbrace{L_{J+1}}_{=[L, L_J]_s} \right]_s &= [L_i, [L, L_J]_s]_s \subseteq \left[ \underbrace{[L_i, L]_s}_{=[L, L_i]_s=L_{i+1}}, L_J \right]_s + \left[ L, \underbrace{[L_i, L_J]_s}_{\substack{\subseteq L_{i+J} \\ \text{(by (8))}}} \right]_s \\
&\quad (\text{by Proposition 4.2 (b)}) \\
&\subseteq \underbrace{[L_{i+1}, L_J]_s}_{\substack{\subseteq L_{(i+1)+J} \\ \text{(by (8), applied to} \\ i+1 \text{ instead of } i)}} + \underbrace{[L, L_{i+J}]_s}_{=L_{i+J+1}=L_{i+(J+1)}} \\
&\subseteq \underbrace{L_{(i+1)+J}}_{=L_{i+(J+1)}} + L_{i+(J+1)} = L_{i+(J+1)} + L_{i+(J+1)} = L_{i+(J+1)}.
\end{aligned}$$

In other words, (7) holds for  $j = J + 1$ . This completes the induction step. Thus, (7) is proven by induction.



Since  $\mathfrak{h} = \bar{\mathfrak{g}} + P$ , we have

$$\left[ \mathfrak{h}, \sum_{i \geq 1} L_i \right]_s = \underbrace{\left[ \bar{\mathfrak{g}}, \sum_{i \geq 1} L_i \right]_s}_{\subseteq \bar{\mathfrak{g}}} + \underbrace{\left[ P, \sum_{i \geq 1} L_i \right]_s}_{\substack{\subseteq \bar{\mathfrak{g}} \\ \text{(by (6))}}} \subseteq \bar{\mathfrak{g}} + \bar{\mathfrak{g}} = \bar{\mathfrak{g}}.$$

Since  $\bar{\mathfrak{g}} \subseteq \sum_{i \geq 1} L_i$ , we now have

$$[\mathfrak{h}, \bar{\mathfrak{g}}]_s \subseteq \left[ \mathfrak{h}, \sum_{i \geq 1} L_i \right]_s \subseteq \bar{\mathfrak{g}}. \quad (9)$$

But we also have  $L = L_1 \subseteq \sum_{i \geq 1} L_i$  and thus

$$[\mathfrak{h}, L]_s \subseteq \left[ \mathfrak{h}, \sum_{i \geq 1} L_i \right]_s \subseteq \bar{\mathfrak{g}}. \quad (10)$$

From this, we easily obtain

$$[\mathfrak{h}, P]_s \subseteq \bar{\mathfrak{g}}$$

<sup>8</sup>.

Now, using  $\mathfrak{h} = \bar{\mathfrak{g}} + P$  again, we obtain

$$[\mathfrak{h}, \mathfrak{h}]_s = \underbrace{[\mathfrak{h}, \bar{\mathfrak{g}}]_s}_{\substack{\subseteq \bar{\mathfrak{g}} \\ \text{(by (9))}}} + \underbrace{[\mathfrak{h}, P]_s}_{\subseteq \bar{\mathfrak{g}}} \subseteq \bar{\mathfrak{g}} + \bar{\mathfrak{g}} = \bar{\mathfrak{g}}.$$

This proves Proposition 5.7 **(a)**.

**(b)** We need to show that  $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}]_s \subseteq \bar{\mathfrak{g}}$ ,  $[\mathfrak{h}, \mathfrak{h}]_s \subseteq \mathfrak{h}$ ,  $[\bar{\mathfrak{g}} + L, \bar{\mathfrak{g}} + L]_s \subseteq \bar{\mathfrak{g}} + L$  and  $[\mathfrak{h} + L, \mathfrak{h} + L]_s \subseteq \mathfrak{h} + L$ .

The relation  $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}]_s \subseteq \bar{\mathfrak{g}}$  follows immediately from  $\left[ \underbrace{\bar{\mathfrak{g}}}_{\subseteq \mathfrak{h}}, \underbrace{\bar{\mathfrak{g}}}_{\subseteq \mathfrak{h}} \right]_s \subseteq [\mathfrak{h}, \mathfrak{h}]_s \subseteq \bar{\mathfrak{g}}$ .

The relation  $[\mathfrak{h}, \mathfrak{h}]_s \subseteq \mathfrak{h}$  follows from  $[\mathfrak{h}, \mathfrak{h}]_s \subseteq \bar{\mathfrak{g}} \subseteq \mathfrak{h}$ .

---

<sup>8</sup>*Proof.* It is clearly enough to show that  $[\mathfrak{h}, xx]_s \subseteq \bar{\mathfrak{g}}$  for every  $x \in L$  (since the  $\mathbf{k}$ -module  $P$  is spanned by elements of the form  $x \otimes x = xx$  for  $x \in L$ ). So let  $x \in L$ . Then,

$$\begin{aligned} [\mathfrak{h}, xx]_s &\subseteq \left[ \left[ \mathfrak{h}, \underbrace{x}_{\in L} \right]_s, \underbrace{x}_{\in L} \right]_s && \text{(by (5))} \\ &\subseteq \left[ \underbrace{[\mathfrak{h}, L]_s}_{\subseteq \bar{\mathfrak{g}}}, L \right]_s \subseteq [\bar{\mathfrak{g}}, L]_s = [L, \bar{\mathfrak{g}}]_s \subseteq \bar{\mathfrak{g}} && \text{(by Proposition 5.6 (a))}, \end{aligned}$$

qed.

We have

$$\begin{aligned} [\mathfrak{h} + L, \mathfrak{h} + L]_s &= \underbrace{[\mathfrak{h}, \mathfrak{h}]_s}_{\subseteq \bar{\mathfrak{g}}} + \underbrace{[\mathfrak{h}, L]_s}_{\substack{\subseteq \bar{\mathfrak{g}} \\ \text{(by (10))}}} + \underbrace{[L, \mathfrak{h}]_s}_{\substack{= [\mathfrak{h}, L]_s \subseteq \bar{\mathfrak{g}} \\ \text{(by (10))}}} + \underbrace{[L, L]_s}_{= L_2 \subseteq L_2 + L_3 + L_4 + \dots = \bar{\mathfrak{g}}} \\ &\subseteq \bar{\mathfrak{g}} + \bar{\mathfrak{g}} + \bar{\mathfrak{g}} + \bar{\mathfrak{g}} = \bar{\mathfrak{g}}. \end{aligned}$$

Thus,  $\left[ \underbrace{\bar{\mathfrak{g}}}_{\subseteq \mathfrak{h}} + L, \underbrace{\bar{\mathfrak{g}}}_{\subseteq \mathfrak{h}} + L \right]_s \subseteq [\mathfrak{h} + L, \mathfrak{h} + L]_s \subseteq \bar{\mathfrak{g}} \subseteq \bar{\mathfrak{g}} + L$  and  $[\mathfrak{h} + L, \mathfrak{h} + L]_s \subseteq \bar{\mathfrak{g}} \subseteq \mathfrak{h} \subseteq \mathfrak{h} + L$ . This proves everything we needed to show. Proposition 5.7 (b) is thus shown.  $\square$

## 6. The kernel of $\mathfrak{t}$

**Definition 6.1.** If  $U$  is any  $\mathbf{k}$ -submodule of a  $\mathbf{k}$ -algebra  $A$ , then we define  $U^i$  to be the  $\mathbf{k}$ -submodule  $\underbrace{UU \cdots U}_{i \text{ times}}$  of  $A$  for every  $i \in \mathbb{N}$ . When  $i = 0$ , this

$\mathbf{k}$ -submodule means the copy of  $\mathbf{k}$  in  $A$  (that is, the  $\mathbf{k}$ -linear span of  $1_A$ ).

If  $U$  is any  $\mathbf{k}$ -submodule of a  $\mathbf{k}$ -algebra  $A$ , then we let  $U^*$  denote the  $\mathbf{k}$ -submodule  $U^0 + U^1 + U^2 + \cdots$  of  $A$ . This is the  $\mathbf{k}$ -subalgebra of  $A$  generated by  $U$ .

The reader should keep in mind that  $U^*$  (the  $\mathbf{k}$ -subalgebra of  $A$  generated by  $U$ ) and  $U^*$  (the dual  $\mathbf{k}$ -module of  $U$ ) are two different things; they are to be distinguished by the shape of the asterisk/star.

**Proposition 6.2.** We have  $\mathfrak{h}^* \subseteq \text{Ker } \mathfrak{t}$ .

*Proof of Proposition 6.2.* We have  $P \subseteq \text{Ker } \mathfrak{t}$  due to Proposition 4.4 (e). Also,  $L_2 = [L, L]_s \subseteq \text{Ker } \mathfrak{t}$  by Proposition 4.4 (c). Using this and Proposition 4.4 (d), we can show that  $L_i \subseteq \text{Ker } \mathfrak{t}$  for each  $i \geq 2$  (by induction over  $i$ ). Thus,  $\bar{\mathfrak{g}} \subseteq \text{Ker } \mathfrak{t}$  (since  $\bar{\mathfrak{g}} = L_2 + L_3 + L_4 + \cdots$ ). Combined with  $P \subseteq \text{Ker } \mathfrak{t}$ , this yields  $\mathfrak{h} \subseteq \text{Ker } \mathfrak{t}$  (since  $\mathfrak{h} = \bar{\mathfrak{g}} + P$ ). Since  $\text{Ker } \mathfrak{t}$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ , this yields that  $\mathfrak{h}^* \subseteq \text{Ker } \mathfrak{t}$ . This proves Proposition 6.2.  $\square$

Our main goal is to prove that the inclusion in Proposition 6.2 actually becomes an equality if the  $\mathbf{k}$ -module  $L$  is free. First, we show three simple lemmas:

**Lemma 6.3.** Let  $u \in L$ . Let  $S$  be a graded  $\mathbf{k}$ -submodule of  $T(L)$  such that  $uu \in S^*$  and  $[u, S]_s \subseteq S^*$ . Then,  $S^* + S^*u$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ .

*Proof of Lemma 6.3.* Clearly,  $1 \in S^* \subseteq S^* + S^*u$ . Hence, it only remains to show that  $(S^* + S^*u)(S^* + S^*u) \subseteq S^* + S^*u$ .

We know that  $S^*$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ ; thus,  $S^*S^* \subseteq S^*$ . Hence,  $S^* \underbrace{uu}_{\in S^*} \subseteq S^*S^* \subseteq S^*$ .

We have

$$uS \subseteq S^* + Su \quad (11)$$

<sup>9</sup>. Now, for every  $i \in \mathbb{N}$ , we have

$$uS^i \subseteq S^* + S^i u \quad (12)$$

<sup>10</sup>. Hence,

$$uS^* \subseteq S^* + S^* u$$

---

<sup>9</sup>*Proof of (11):* It clearly suffices to show that  $us \in Su + S^*$  for every  $s \in S$ . So let us fix some  $s \in S$ . We can WLOG assume that  $s$  is homogeneous (since  $S$  is graded). Assume this. Then,  $u \in L^{\otimes n}$  for some  $n \in \mathbb{N}$ . Consider this  $n$ . Thus,  $[u, s]_s = us - (-1)^{1 \cdot n} su$ , so that

$$us = \left[ u, \underbrace{s}_{\in S} \right]_s + \underbrace{(-1)^{1 \cdot n} s u}_{\in S} \in \underbrace{[u, S]_s}_{\subseteq S^*} + Su \subseteq S^* + Su.$$

This proves (11).

<sup>10</sup>*Proof of (12):* We will prove (12) by induction over  $i$ :

*Induction base:* We have  $u \underbrace{S^0}_{=\mathbf{k}} = u\mathbf{k} = \underbrace{\mathbf{k}}_{=S^0} u = S^0 u \subseteq S^* + S^0 u$ . In other words, (12) holds

for  $i = 0$ . This completes the induction base.

*Induction step:* Let  $I \in \mathbb{N}$ . Assume that (12) is proven for  $i = I$ . We need to prove (12) for  $i = I + 1$ .

We know that (12) is proven for  $i = I$ . In other words,  $uS^I \subseteq S^* + S^I u$ . Thus,

$$\begin{aligned} u \underbrace{S^{I+1}}_{=S^I S} &= \underbrace{uS^I}_{\subseteq S^* + S^I u} S \subseteq (S^* + S^I u) S \subseteq \underbrace{S^* S}_{\subseteq S^*} + S^I \underbrace{uS}_{\subseteq S^* + Su \text{ (by (11))}} \\ &\subseteq S^* + S^I (S^* + Su) \subseteq S^* + \underbrace{S^I S^*}_{\subseteq S^*} + \underbrace{S^I S}_{=S^{I+1}} u \subseteq \underbrace{S^* + S^*}_{=S^*} + S^{I+1} u = S^* + S^{I+1} u. \end{aligned}$$

In other words, (12) holds for  $i = I + 1$ . This completes the induction step, and thus (12) is proven.

<sup>11</sup>. Thus,

$$\begin{aligned}
u(S^* + S^*u) &= \underbrace{uS^*}_{\subseteq S^* + S^*u} + \underbrace{uS^*}_{\subseteq S^* + S^*u} u \\
&\subseteq (S^* + S^*u) + (S^* + S^*u)u = S^* + S^*u + S^*u + \underbrace{S^*uu}_{\subseteq S^*} \\
&\subseteq S^* + S^*u + S^*u + S^* = S^* + S^*u,
\end{aligned}$$

so that

$$\begin{aligned}
&(S^* + S^*u)(S^* + S^*u) \\
&= S^*(S^* + S^*u) + \underbrace{S^*u(S^* + S^*u)}_{\subseteq S^* + S^*u} \\
&\subseteq S^*(S^* + S^*u) + S^*(S^* + S^*u) = S^*(S^* + S^*u) = \underbrace{S^*S^*}_{\subseteq S^*} + \underbrace{S^*S^*u}_{\subseteq S^*} \subseteq S^* + S^*u,
\end{aligned}$$

and this completes our proof of Lemma 6.3.  $\square$

**Lemma 6.4.** Let  $N$  be a graded  $\mathbf{k}$ -submodule of  $T(L)$ . Let  $g \in L^*$  be such that  $\partial_g(N) = 0$ . Then,  $\partial_g(N^*) = 0$ .

*Proof of Lemma 6.4.* First, we claim that

$$\partial_g(N^i) = 0 \quad (13)$$

for all  $i \in \mathbb{N}$ .

*Proof of (13).* We shall prove (13) by induction over  $i$ :

*Induction base:* Proposition 3.2 (a) yields  $\partial_g(1) = 0$ . Thus,  $\partial_g(N^0) = 0$  (since the  $\mathbf{k}$ -module  $N^0$  is spanned by 1). In other words, (13) holds for  $i = 0$ . This completes the induction base.

*Induction step:* Let  $I \in \mathbb{N}$ . Assume that (13) holds for  $i = I$ . We now need to show that (13) holds for  $i = I + 1$ .

In other words, we need to show that  $\partial_g(N^{I+1}) = 0$ . For this, it is clearly enough to prove that  $\partial_g(UV) = 0$  for all  $U \in N$  and  $V \in N^I$  (since the  $\mathbf{k}$ -module  $N^{I+1} = NN^I$  is spanned by elements of the form  $UV$  with  $U \in N$

---

<sup>11</sup>since  $S^* = S^0 + S^1 + S^2 + \dots = \sum_{i \in \mathbb{N}} S^i$  and thus

$$\begin{aligned}
u \underbrace{S^*}_{= \sum_{i \in \mathbb{N}} S^i} &= u \left( \sum_{i \in \mathbb{N}} S^i \right) = \sum_{i \in \mathbb{N}} \underbrace{uS^i}_{\subseteq S^* + S^i u \text{ (by (12))}} \subseteq \sum_{i \in \mathbb{N}} (S^* + S^i u) \\
&\subseteq S^* + \sum_{i \in \mathbb{N}} S^i u = S^* + \underbrace{\left( \sum_{i \in \mathbb{N}} S^i \right)}_{= S^*} u = S^* + S^*u
\end{aligned}$$

and  $V \in N^I$ ). So let  $U \in N$  and  $V \in N^I$ . Since both  $N$  and  $N^I$  are graded  $\mathbf{k}$ -modules (because  $N$  is graded), we can WLOG assume that  $U$  and  $V$  are homogeneous elements of  $T(L)$ . Assume this, and let  $n \in \mathbb{N}$  and  $m \in \mathbb{N}$  be such that  $U \in L^{\otimes n}$  and  $V \in L^{\otimes m}$ . Proposition 3.2 (b) (applied to  $a = U$  and  $b = V$ ) then yields  $\partial_g(UV) = \partial_g(U)V + (-1)^n U\partial_g(V)$ . But we assumed that (13) holds for  $i = I$ . In other words,  $\partial_g(N^I) = 0$ , so that  $\partial_g(V) = 0$  (since  $V \in N^I$ ). Also,  $\partial_g(N) = 0$  and thus  $\partial_g(U) = 0$  (since  $U \in N$ ). Now,  $\partial_g(UV) = \underbrace{\partial_g(U)}_{=0}V + (-1)^n U\underbrace{\partial_g(V)}_{=0} = 0$ , which is exactly what we wanted to prove. Thus, the induction step is complete, and (13) is proven.

Now, the definition of  $N^*$  yields  $N^* = N^0 + N^1 + N^2 + \dots = \sum_{i \in \mathbb{N}} N^i$ , so that

$$\partial_g(N^*) = \partial_g\left(\sum_{i \in \mathbb{N}} N^i\right) = \sum_{i \in \mathbb{N}} \underbrace{\partial_g(N^i)}_{\substack{=0 \\ \text{(by (13))}}} = 0. \text{ This proves Lemma 6.4. } \quad \square$$

**Lemma 6.5.** Let  $M$  be a  $\mathbf{k}$ -submodule of  $L$ . Let  $g \in L^*$  be such that  $g(M) = 0$ . Then:

(a) We have  $\partial_g((M + \mathfrak{h})^*) = 0$ .

(b) Let  $q \in L$  be such that  $g(q) = 1$ . Let  $U_0$  and  $U_1$  be any two elements of  $(M + \mathfrak{h})^*$ . If  $\partial_g(U_0 + U_1q) = 0$ , then  $U_1 = 0$ .

*Proof of Lemma 6.5.* The  $\mathbf{k}$ -submodules  $M$  and  $\mathfrak{h}$  of  $T(L)$  are graded (for  $M$ , this is because  $M \subseteq L \subseteq L^{\otimes 1}$ ). Thus,  $M + \mathfrak{h}$  is graded.

Proposition 6.2 yields  $\mathfrak{h}^* \subseteq \text{Ker } \mathbf{t}$ , so that  $\mathfrak{h} \subseteq \mathfrak{h}^* \subseteq \text{Ker } \mathbf{t}$  and therefore

$$\partial_g\left(\underbrace{\mathfrak{h}}_{\subseteq \text{Ker } \mathbf{t}}\right) \subseteq \partial_g(\text{Ker } \mathbf{t}) = 0 \text{ (by Proposition 3.3 (a)). Thus, } \partial_g(\mathfrak{h}) = 0. \text{ Also,}$$

$\partial_g(v) = g(v)$  holds for every  $v \in L$  (by the definition of  $\partial_g$ ), and thus, in particular, for every  $v \in M$ . Hence,  $\partial_g(M) = g(M) = 0$ . Now,

$$\partial_g(M + \mathfrak{h}) = \underbrace{\partial_g(M)}_{=0} + \underbrace{\partial_g(\mathfrak{h})}_{=0} = 0.$$

Hence, Lemma 6.4 (applied to  $N = M + \mathfrak{h}$ ) yields  $\partial_g((M + \mathfrak{h})^*) = 0$ . This proves Lemma 6.5 (a).

(b) We have  $\partial_g(v) = g(v)$  for every  $v \in L$  (by the definition of  $\partial_g$ ). Applied to  $v = q$ , this yields  $\partial_g(q) = g(q) = 1$ .

We have  $U_0 \in (M + \mathfrak{h})^*$  and thus  $\partial_g(U_0) = 0$  (since  $\partial_g((M + \mathfrak{h})^*) = 0$ ). Similarly,  $\partial_g(U_1) = 0$ .

Let  $\mathbf{s}$  be the  $\mathbf{k}$ -module homomorphism  $T(L) \rightarrow T(L)$  which is given by

$$\mathbf{s}(U) = (-1)^n U \quad \text{for any } n \in \mathbb{N} \text{ and any } U \in L^{\otimes n}.$$

Clearly, this map  $\mathbf{s}$  is an isomorphism of graded  $\mathbf{k}$ -modules. (It is actually an involution and an isomorphism of graded  $\mathbf{k}$ -algebras.)

It is easy to see that any  $a \in T(L)$  and  $b \in T(L)$  satisfy  $\partial_g(ab) = \partial_g(a)b + \mathbf{s}(a)\partial_g(b)$ .<sup>12</sup> Applying this to  $a = U_1$  and  $b = q$ , we obtain  $\partial_g(U_1q) = \underbrace{\partial_g(U_1)}_{=0}q + \mathbf{s}(U_1)\underbrace{\partial_g(q)}_{=1} = 0 + \mathbf{s}(U_1) = \mathbf{s}(U_1)$ .

Now, assume that  $\partial_g(U_0 + U_1q) = 0$ . Thus,  $0 = \partial_g(U_0 + U_1q) = \underbrace{\partial_g(U_0)}_{=0} + \underbrace{\partial_g(U_1q)}_{=\mathbf{s}(U_1)} = \mathbf{s}(U_1)$ . Since  $\mathbf{s}$  is an isomorphism, this yields  $0 = U_1$ . This proves Lemma 6.5 (b).  $\square$

The following is our main result:

■ **Theorem 6.6.** Assume that the  $\mathbf{k}$ -module  $L$  is free. Then,  $\mathfrak{h}^* = \text{Ker } \mathbf{t}$ .

*Proof of Theorem 6.6.* Proposition 6.2 shows that  $\mathfrak{h}^* \subseteq \text{Ker } \mathbf{t}$ . We thus only need to verify that  $\text{Ker } \mathbf{t} \subseteq \mathfrak{h}^*$ . This means proving that every  $U \in \text{Ker } \mathbf{t}$  satisfies  $U \in \mathfrak{h}^*$ . So let us fix  $U \in \text{Ker } \mathbf{t}$ .

We know that the  $\mathbf{k}$ -module  $L$  is free; it thus has a basis. Since the tensor  $U \in T(L)$  can be constructed using only finitely many elements of this basis, we can thus WLOG assume that the basis of  $L$  is finite. Let us assume this, and let us denote this basis by  $(e_1, e_2, \dots, e_n)$ .

For every  $i \in \{1, 2, \dots, n\}$ , let  $e_i^* : L \rightarrow \mathbf{k}$  be the  $\mathbf{k}$ -linear map which sends  $e_i$  to 1 and sends every other  $e_j$  to 0.

For every  $k \in \{0, 1, \dots, n\}$ , we let  $M_k$  denote the  $\mathbf{k}$ -submodule of  $L$  spanned by  $e_1, e_2, \dots, e_k$ . Thus,  $M_0 = 0$  and  $M_n = L$ . Clearly, every  $k \in \{1, 2, \dots, n\}$  satisfies

$$M_k = M_{k-1} + \mathbf{k}e_k. \quad (14)$$

For every  $k \in \{0, 1, \dots, n\}$ , we set  $\mathfrak{h}_k = M_k + \mathfrak{h}$  and  $H_k = \mathfrak{h}_k^*$ .

Notice that  $\mathfrak{h}_n = M_n + \mathfrak{h} \supseteq M_n = L$  and thus  $H_n = \mathfrak{h}_n^* \supseteq L^* = T(L)$ . Hence,  $H_n = T(L)$ . Now,  $U \in T(L) = H_n$ .

On the other hand, the definition of  $\mathfrak{h}_0$  yields  $\mathfrak{h}_0 = \underbrace{M_0}_{=0} + \mathfrak{h} = \mathfrak{h}$  and thus

$$H_0 = \mathfrak{h}_0^* = \mathfrak{h}^*.$$

We shall now prove that every  $k \in \{1, 2, \dots, n\}$  satisfies the following implication:

$$\text{if } U \in H_k, \text{ then } U \in H_{k-1}. \quad (15)$$

Once this is proven, we will be able to argue that  $U \in H_n$  (as we know), thus  $U \in H_{n-1}$  (by (15)), thus  $U \in H_{n-2}$  (by (15) again), and so on – until we finally arrive at  $U \in H_0$ . Since  $H_0 = \mathfrak{h}^*$ , this rewrites as  $U \in \mathfrak{h}^*$ , and thus we are done.

Therefore, it only remains to prove (15). So let us fix  $k \in \{1, 2, \dots, n\}$ , and assume that  $U \in H_k$ . We now need to show that  $U \in H_{k-1}$ .

---

<sup>12</sup>Indeed, it is sufficient to prove this when  $a$  is homogeneous, but in this case it follows from Proposition 3.2 (b).

We first notice that  $\mathfrak{h}_{k-1}$  is a graded  $\mathbf{k}$ -submodule of  $T(L)$  (by its definition, since  $M_k$  and  $\mathfrak{h}$  are graded  $\mathbf{k}$ -submodules). Furthermore,  $e_k e_k = e_k \otimes e_k \in P$  (by the definition of  $P$ ), and thus  $e_k e_k \in P \subseteq \mathfrak{h} \subseteq M_{k-1} + \mathfrak{h} = \mathfrak{h}_{k-1} \subseteq \mathfrak{h}_{k-1}^*$ . Moreover,

$$\begin{aligned} [e_k, \mathfrak{h}_{k-1}]_s &= \left[ \underbrace{e_k}_{\in L}, \underbrace{M_{k-1}}_{\subseteq L} \right]_s + \left[ \underbrace{e_k}_{\in L}, \mathfrak{h} \right]_s \quad (\text{since } \mathfrak{h}_{k-1} = M_{k-1} + \mathfrak{h}) \\ &\subseteq \underbrace{[L, L]_s}_{=L_2 \subseteq L_2 + L_3 + L_4 + \dots = \overline{\mathfrak{g}}} + \underbrace{[L, \mathfrak{h}]_s}_{\subseteq \overline{\mathfrak{g}}} \quad (\text{by Proposition 5.6 (b)}) \\ &\subseteq \overline{\mathfrak{g}} + \overline{\mathfrak{g}} = \overline{\mathfrak{g}} \subseteq \mathfrak{h} \\ &\subseteq M_{k-1} + \mathfrak{h} = \mathfrak{h}_{k-1} \subseteq \mathfrak{h}_{k-1}^*. \end{aligned}$$

Thus, Lemma 6.3 (applied to  $u = e_k$  and  $S = \mathfrak{h}_{k-1}$ ) yields that  $\mathfrak{h}_{k-1}^* + \mathfrak{h}_{k-1}^* e_k$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ . In other words,  $H_{k-1} + H_{k-1} e_k$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$  (since  $H_{k-1} = \mathfrak{h}_{k-1}^*$ ). This  $\mathbf{k}$ -subalgebra contains  $\mathfrak{h}_k$  as a subset<sup>13</sup>, and thus we have  $H_k \subseteq H_{k-1} + H_{k-1} e_k$ <sup>14</sup>. (Actually,  $H_k = H_{k-1} + H_{k-1} e_k$ , but we don't need this.)

Now,  $U \in H_k \subseteq H_{k-1} + H_{k-1} e_k$ . Therefore, there exist two elements  $U_0$  and  $U_1$  of  $H_{k-1}$  such that  $U = U_0 + U_1 e_k$ . Consider these  $U_0$  and  $U_1$ . We have

$$\partial_{e_k}^* \left( \underbrace{U_0 + U_1 e_k}_{=U} \right) = \partial_{e_k}^* \left( \underbrace{U}_{\in \text{Ker } \mathbf{t}} \right) \in \partial_{e_k}^* (\text{Ker } \mathbf{t}) = 0$$

(by Proposition 3.3 (a), applied to  $g = e_k^*$ ), so that  $\partial_{e_k}^* (U_0 + U_1 e_k) = 0$ .

The elements  $U_0$  and  $U_1$  belong to  $H_{k-1} = \mathfrak{h}_{k-1}^* = (M_{k-1} + \mathfrak{h})^*$  (since  $\mathfrak{h}_{k-1} = M_{k-1} + \mathfrak{h}$ ). We can thus apply Lemma 6.5 (b) to  $M = M_{k-1}$ ,  $g = e_k^*$  and  $q = e_k$  (since  $e_k^* (M_{k-1}) = 0$  (because  $M_{k-1}$  is spanned by  $e_1, e_2, \dots, e_{k-1}$ ) and  $e_k^* (e_k) = 1$  and  $\partial_{e_k}^* (U_0 + U_1 e_k) = 0$ ). As a result, we see that  $U_1 = 0$ . Thus,  $U = U_0 + \underbrace{U_1}_{=0} e_k = U_0 \in H_{k-1}$ . This completes the proof of (15). As we already mentioned,

this finishes the proof of Theorem 6.6.  $\square$

<sup>13</sup>Proof. We have  $\mathfrak{h}_k = M_k + \mathfrak{h}$  and similarly  $\mathfrak{h}_{k-1} = M_{k-1} + \mathfrak{h}$ . Thus,

$$\begin{aligned} \mathfrak{h}_k &= \underbrace{M_k}_{=M_{k-1} + \mathbf{k}e_k \text{ (by (14))}} + \mathfrak{h} = M_{k-1} + \mathbf{k}e_k + \mathfrak{h} = \underbrace{M_{k-1} + \mathfrak{h}}_{=\mathfrak{h}_{k-1} \subseteq \mathfrak{h}_{k-1}^* = H_{k-1} \text{ (since } H_{k-1} \text{ was defined as } \mathfrak{h}_{k-1}^*)} + \underbrace{\mathbf{k}}_{\subseteq H_{k-1}} e_k \\ &\subseteq H_{k-1} + H_{k-1} e_k, \end{aligned}$$

qed.

<sup>14</sup>Proof. The  $\mathbf{k}$ -subalgebra  $H_{k-1} + H_{k-1} e_k$  of  $T(L)$  contains  $\mathfrak{h}_k$  as a subset. Hence, it also contains  $\mathfrak{h}_k^*$  as a subset (since  $\mathfrak{h}_k^*$  is the  $\mathbf{k}$ -subalgebra of  $T(L)$  generated by  $\mathfrak{h}_k$ ). In other words,  $H_{k-1} + H_{k-1} e_k \supseteq \mathfrak{h}_k^* = H_k$ , qed.

Our idea to prove (15) goes back to Specht; it is, in some sense, an analogue of the argument [Specht50, VI, Zweiter Schritt] where he gradually moves the variables  $x_1, x_2, \dots, x_n$  (which can be roughly seen as corresponding to our basis vectors  $e_1, e_2, \dots, e_n$ ) inside commutators.

**Remark 6.7.** We suspect that Theorem 6.6 still holds if we replace the word “free” by “flat”. (We furthermore suspect that Lazard’s theorem might help prove this.) However, Theorem 6.6 does not hold if we completely remove the condition that  $L$  be free. A counterexample (one in which  $L^{\otimes 2} \cap \text{Ker } \mathbf{t} \not\subseteq \mathfrak{h}^*$ ) can be obtained from [Lundkv08, Example 4.6]. (Indeed, it is not hard to see that  $L^{\otimes 2} \cap \text{Ker } \mathbf{t} \subseteq \mathfrak{h}^*$  holds if and only if, using the notations of [Lundkv08], the canonical map  $\Gamma_{\mathbf{k}}^2(L) \rightarrow \text{TS}_{\mathbf{k}}^2(L)$  is surjective. And [Lundkv08, Example 4.6] shows that the latter does not always hold.)

## 7. The even analogue

The map  $\mathbf{t}$  we introduced in Definition 2.3 has a natural analogue, which is obtained by removing the  $(-1)^{i-1}$  signs from its definition. (One might even argue that this analogue is more natural than  $\mathbf{t}$ ; at any rate, it is more directly related to both the random-to-top shuffling operator and Specht’s construction.) We shall denote this analogue by  $\mathbf{t}'$ ; here is its precise definition:

**Definition 7.1.** Let  $\mathbf{t}' : T(L) \rightarrow T(L)$  be the  $\mathbf{k}$ -linear map which acts on pure tensors according to the formula

$$\mathbf{t}'(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \sum_{i=1}^k u_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u}_i \otimes \cdots \otimes u_k$$

(for all  $k \in \mathbb{N}$  and  $u_1, u_2, \dots, u_k \in L$ ). (This is clearly well-defined.) Thus,  $\mathbf{t}'$  is a graded  $\mathbf{k}$ -module endomorphism of  $T(L)$ .

Those familiar with superalgebras will immediately notice that the maps  $\mathbf{t}$  and  $\mathbf{t}'$  can be seen as two particular cases of one single unifying construction (a map defined on the tensor algebra of a  $\mathbf{k}$ -supermodule, which, roughly speaking, differs from  $\mathbf{t}$  in that the sign  $(-1)^{i-1}$  is replaced by  $(-1)^{(\deg u_1 + \deg u_2 + \cdots + \deg u_{i-1})(\deg u_i)}$ ). We shall not follow this lead, but rather study the map  $\mathbf{t}'$  separately. Unlike for the map  $\mathbf{t}$ , I am not aware of a single general description of  $\text{Ker}(\mathbf{t}')$  that works with no restrictions on  $\mathbf{k}$  (whenever  $L$  is a free  $\mathbf{k}$ -module). However, I can describe  $\text{Ker}(\mathbf{t}')$  when the additive group  $\mathbf{k}$  is torsionfree and when  $\mathbf{k}$  is an  $\mathbb{F}_p$ -algebra for some prime  $p$  (of course, in both cases,  $L$  still has to be a free  $\mathbf{k}$ -module). The answers in these two cases are different, and there does not seem to be an obvious way to extend the argument to cases such as  $\mathbf{k} = \mathbb{Z}/6\mathbb{Z}$ .

Let us first present analogues of some objects we constructed earlier in our study of  $\mathbf{t}$ . First, here is an analogue of Definition 3.1:



**Definition 7.2.** Let  $L^*$  denote the dual  $\mathbf{k}$ -module  $\text{Hom}(L, \mathbf{k})$  of  $L$ . If  $g \in L^*$ , then we define a  $\mathbf{k}$ -linear map  $\partial'_g : T(L) \rightarrow T(L)$  by

$$\partial'_g(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \sum_{i=1}^k g(u_i) \cdot u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_k$$

for all  $k \in \mathbb{N}$  and  $u_1, u_2, \dots, u_k \in L$ . (Again, it is easy to check that this is well-defined.)

The following proposition is an analogue of Proposition 3.2. (However, unlike Proposition 3.2, it does not require  $a$  to be homogeneous, since there are no more signs that could change depending on its degree.)

**Proposition 7.3.** Let  $g \in L^*$ .

(a) Then,  $\partial'_g(1) = 0$ .

(b) Also, if  $a \in T(L)$  and  $b \in T(L)$ , then  $\partial'_g(ab) = \partial'_g(a)b + a\partial'_g(b)$ .

Of course, Proposition 7.3 (b) says precisely that  $\partial'_g$  is a derivation  $T(L) \rightarrow T(L)$ .

Next comes the analogue of Proposition 3.3:

**Proposition 7.4.** (a) We have  $\partial'_g(\text{Ker}(\mathbf{t}')) = 0$  for every  $g \in L^*$ .

(b) Assume that  $L$  is a free  $\mathbf{k}$ -module. Then,

$$\text{Ker}(\mathbf{t}') = \left\{ U \in T(L) \mid \partial'_g(U) = 0 \text{ for every } g \in L^* \right\}.$$

*Proof of Proposition 7.4.* The proof of Proposition 7.4 is analogous to that of Proposition 3.3.  $\square$

The analogue of the supercommutator  $[\cdot, \cdot]_s$  is the plain commutator  $[\cdot, \cdot]$ , which is defined by  $[U, V] = UV - VU$  for any  $U \in T(L)$  and  $V \in T(L)$ . Again, this analogue is less troublesome to work with than the supercommutator  $[\cdot, \cdot]_s$  because there is no dependence on the degrees of  $U$  and  $V$  in its definition.

For the sake of completeness, we state an analogue of Proposition 4.2 (which is really well-known):

**Proposition 7.5.** Let  $U \in T(L)$ ,  $V \in T(L)$  and  $W \in T(L)$ . Then:

(a) We have  $[U, VW] = [U, V]W + V[U, W]$ .

(b) We have  $[U, [V, W]] = [[U, V], W] + [V, [U, W]]$ .

Next, we formulate an analogue to Proposition 4.4:

**Proposition 7.6. (a)** We have  $L^{\otimes 0} \subseteq \text{Ker}(\mathbf{t}')$ .

**(b)** We have  $\text{Ker}(\mathbf{t}') \cdot \text{Ker}(\mathbf{t}') \subseteq \text{Ker}(\mathbf{t}')$ .

**(c)** We have  $[L, L] \subseteq \text{Ker}(\mathbf{t}')$ .

**(d)** We have  $[L, \text{Ker}(\mathbf{t}')] \subseteq \text{Ker}(\mathbf{t}')$ .

Notice that Proposition 7.6 has no part **(e)**, unlike Proposition 4.4. Indeed, there is no analogue to Proposition 4.4 **(e)** for the map  $\mathbf{t}'$  in the general case. (We will later see something that can be regarded as an analogue in the positive-characteristic case.)

We next define an analogue to the  $\mathbf{k}$ -submodules  $L_1, L_2, L_3, \dots$ :

**Definition 7.7.** We recursively define a sequence  $(L'_1, L'_2, L'_3, \dots)$  of  $\mathbf{k}$ -submodules of  $T(L)$  as follows: We set  $L'_1 = L$ , and  $L'_{i+1} = [L, L'_i]$  for every positive integer  $i$ .

For instance,  $L'_2 = [L, L]$  and  $L'_3 = [L, L'_2] = [L, [L, L]]$ .

The  $\mathbf{k}$ -submodule  $L'_1 + L'_2 + L'_3 + \dots$  of  $T(L)$  is a Lie subalgebra of  $T(L)$ . When  $L$  is a free  $\mathbf{k}$ -module, this Lie subalgebra is isomorphic to the free Lie algebra on  $L$ .

Of course,  $L'_i \subseteq L^{\otimes i}$  for every positive integer  $i$ .

The analogue to  $\bar{\mathfrak{g}}$  is what you would expect:

**Definition 7.8.** Let  $\bar{\mathfrak{g}}'$  denote the  $\mathbf{k}$ -submodule  $L'_2 + L'_3 + L'_4 + \dots$  of  $T(L)$ .

The analogue of  $P$  is more interesting – in that it is the zero module  $0 \subseteq T(L)$ . At least if we don't make any assumptions on  $\mathbf{k}$ , this is the most reasonable choice we could make for the analogue of  $P$ . (Later, in the positive-characteristic case, we shall encounter a more interesting  $\mathbf{k}$ -submodule similar to  $P$ .)

The analogue of  $\mathfrak{h}$ , so far, has to be  $\bar{\mathfrak{g}}'$  (since the analogue of  $P$  is 0). There is no analogue of Proposition 5.4. We have an analogue of Proposition 5.6 **(a)**, however:

**Proposition 7.9.** We have  $[L, \bar{\mathfrak{g}}'] \subseteq \bar{\mathfrak{g}}'$ .

*Proof of Proposition 7.9.* This is proven in the same way as Proposition 5.6 **(a)**.  $\square$

Here is an analogue of parts of Proposition 5.7:

**Proposition 7.10. (a)** We have  $[\bar{\mathfrak{g}}', \bar{\mathfrak{g}}'] \subseteq \bar{\mathfrak{g}}'$ .

**(b)** The two  $\mathbf{k}$ -submodules  $\bar{\mathfrak{g}}'$  and  $\bar{\mathfrak{g}}' + L$  of  $T(L)$  are invariant under the commutator  $[\cdot, \cdot]$ . (In other words, they are Lie subalgebras of  $T(L)$  (with the commutator  $[\cdot, \cdot]$  as the Lie bracket).)

*Proof of Proposition 7.10.* This is analogous to the relevant parts of the proof of Proposition 5.7. (The  $\mathbf{k}$ -submodule  $\bar{\mathfrak{g}}'$  takes the roles of both  $\bar{\mathfrak{g}}$  and  $\mathfrak{h}$ , and the zero module 0 takes the role of  $P$ .)  $\square$

We can now state the analogue of Proposition 6.2:

■ **Proposition 7.11.** We have  $(\bar{\mathfrak{g}}')^* \subseteq \text{Ker}(\mathfrak{t}')$ .

*Proof of Proposition 7.11.* Unsurprisingly, this is analogous to the proof of Proposition 6.2.  $\square$

What is not straightforward is finding the right analogue of Lemma 6.3. We must no longer assume  $uu \in S^*$  (since this won't be satisfied in the situation we are going to apply this lemma to). Thus, a two-term sum like the  $S^* + S^*u$  in Lemma 6.3 won't work anymore; instead we need an infinite sum  $S^* + S^*u + S^*u^2 + \dots = \sum_{j \in \mathbb{N}} S^*u^j$ .<sup>15</sup> Here is the exact statement:

■ **Lemma 7.12.** Let  $u \in L$ . Let  $S$  be a  $\mathbf{k}$ -submodule of  $T(L)$  such that  $[u, S] \subseteq S^*$ . Then,  $\sum_{j \in \mathbb{N}} S^*u^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ .

*Proof of Lemma 7.12.* We have  $uS^* \subseteq S^* + S^*u$ . (This can be proven just as in the proof of Lemma 6.3, mutatis mutandis.) Now,

$$u^j S^* \subseteq \sum_{k=0}^j S^* u^k \quad \text{for every } j \in \mathbb{N} \quad (16)$$

---

<sup>15</sup>This is similar to the difference between the standard basis vectors of the exterior algebra and the symmetric algebra of a free  $\mathbf{k}$ -module: the former have every  $e_i$  appear at most once, while the latter can have it multiple times.

<sup>16</sup>. Now,

$$\begin{aligned}
& \left( \sum_{j \in \mathbb{N}} S^* u^j \right) \left( \sum_{j \in \mathbb{N}} S^* u^j \right) \\
&= \left( \sum_{j \in \mathbb{N}} S^* u^j \right) \left( \sum_{i \in \mathbb{N}} S^* u^i \right) = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} S^* \underbrace{u^j S^*}_{\substack{\subseteq \sum_{k=0}^j S^* u^k \\ \text{(by (16))}}} u^i \\
&\subseteq \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} S^* \left( \sum_{k=0}^j S^* u^k \right) u^i = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \sum_{k=0}^j \underbrace{S^* S^*}_{\substack{\subseteq S^* \\ \text{(since } S^* \text{ is a} \\ \text{k-subalgebra)}}} \underbrace{u^k u^i}_{=u^{k+i}} \\
&\subseteq \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \sum_{k=0}^j S^* u^{k+i} \subseteq \sum_{\ell \in \mathbb{N}} S^* u^\ell \quad (\text{since } k+i \in \mathbb{N} \text{ for all } i \in \mathbb{N} \text{ and } k \in \{0, 1, \dots, j\}) \\
&= \sum_{j \in \mathbb{N}} S^* u^j.
\end{aligned}$$

---

<sup>16</sup>*Proof of (16):* We shall prove (16) by induction over  $j$ .

*Induction base:* When  $j = 0$ , the relation (16) rewrites as  $u^0 S^* \subseteq \sum_{k=0}^0 S^* u^k$ . But this is obvious, since  $\underbrace{u^0}_{=1} S^* = S^*$  and  $\sum_{k=0}^0 S^* u^k = S^* \underbrace{u^0}_{=1} = S^*$ . Thus, the relation (16) holds for  $j = 0$ . The induction base is thus complete.

*Induction step:* Let  $J \in \mathbb{N}$ . Assume that (16) holds for  $j = J$ . We now need to prove that (16) holds for  $j = J + 1$ .

We know that (16) holds for  $j = J$ . In other words,  $u^J S^* \subseteq \sum_{k=0}^J S^* u^k$ . Now,

$$\begin{aligned}
\underbrace{u^{J+1} S^*}_{=u u^J} &= u \underbrace{u^J S^*}_{\subseteq \sum_{k=0}^J S^* u^k} \subseteq u \sum_{k=0}^J S^* u^k = \sum_{k=0}^J \underbrace{u S^*}_{\subseteq S^* + S^* u} u^k \\
&\subseteq \sum_{k=0}^J \underbrace{(S^* + S^* u) u^k}_{=S^* u^k + S^* u u^k} = \sum_{k=0}^J (S^* u^k + S^* u u^k) \\
&= \sum_{k=0}^J S^* u^k + \sum_{k=0}^J S^* \underbrace{u u^k}_{=u^{k+1}} = \sum_{k=0}^J S^* u^k + \sum_{k=0}^J S^* u^{k+1} \\
&= \sum_{k=0}^J S^* u^k + \sum_{k=1}^{J+1} S^* u^k = \sum_{k=0}^{J+1} S^* u^k.
\end{aligned}$$

In other words, (16) holds for  $j = J + 1$ . This completes the induction step. Thus, (16) is proven.

Combined with  $1 \in S^* = S^* \underbrace{1}_{=u^0} = S^*u^0 \subseteq \sum_{j \in \mathbb{N}} S^*u^j$ , this yields that  $\sum_{j \in \mathbb{N}} S^*u^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ . This proves Lemma 7.12.  $\square$

Our next lemma is a straightforward analogue of Lemma 6.4:

**Lemma 7.13.** Let  $N$  be a  $\mathbf{k}$ -submodule of  $T(L)$ . Let  $g \in L^*$  be such that  $\partial'_g(N) = 0$ . Then,  $\partial'_g(N^*) = 0$ .

*Proof of Lemma 7.13.* The proof is analogous to that of Lemma 6.4.  $\square$

Next, we state an analogue of Lemma 6.5:

**Lemma 7.14.** Let  $M$  be a  $\mathbf{k}$ -submodule of  $L$ . Let  $g \in L^*$  be such that  $g(M) = 0$ . Then:

(a) We have  $\partial'_g((M + \overline{g}')^*) = 0$ .

(b) Assume that the additive group  $T(L)$  is torsionfree. Let  $q \in L$  be such that  $g(q) = 1$ . Let  $(U_0, U_1, U_2, \dots)$  be a sequence of elements of  $(M + \overline{g}')^*$  such that all but finitely many  $i \in \mathbb{N}$  satisfy  $U_i = 0$ . If  $\partial'_g\left(\sum_{i \in \mathbb{N}} U_i q^i\right) = 0$ , then every positive integer  $i$  satisfies  $U_i = 0$ .

*Proof of Lemma 7.14.* The proof of Lemma 7.14 (a) is analogous to that of Lemma 6.5 (a). It remains to prove part (b).

We assume that the additive group  $T(L)$  is torsionfree. Let  $q \in L$  be such that  $g(q) = 1$ .

In order to prepare for this proof, we shall make a definition. Given an  $N \in \mathbb{N}$ , we say that the sequence  $(U_0, U_1, U_2, \dots)$  of elements of  $(M + \overline{g}')^*$  is  $N$ -supported if every integer  $i \geq N$  satisfies  $U_i = 0$ . Of course, the sequence  $(U_0, U_1, U_2, \dots)$  of elements of  $(M + \overline{g}')^*$  must be  $N$ -supported for some  $N \in \mathbb{N}$  (since all but finitely many  $i \in \mathbb{N}$  satisfy  $U_i = 0$ ). Hence, in order to prove Lemma 7.14 (b), it suffices to show that, for every  $N \in \mathbb{N}$ ,

(Lemma 7.14 (b) holds whenever the sequence  $(U_0, U_1, U_2, \dots)$  is  $N$ -supported). (17)

We shall now prove (17) by induction over  $N$ :

*Induction base:* The only 0-supported sequence  $(U_0, U_1, U_2, \dots)$  is  $(0, 0, 0, \dots)$ . Lemma 7.14 (b) clearly holds for this sequence. Thus, (17) holds for  $N = 0$ . This completes the induction base.

*Induction step:* Fix  $n \in \mathbb{N}$ . Assume that (17) is proven for  $N = n$ . We now need to prove (17) for  $N = n + 1$ .

We assumed that (17) is proven for  $N = n$ . In other words,

(Lemma 7.14 (b) holds whenever the sequence  $(U_0, U_1, U_2, \dots)$  is  $n$ -supported). (18)

Let  $(U_0, U_1, U_2, \dots)$  be a sequence of elements of  $(M + \overline{g}')^*$  such that all but finitely many  $i \in \mathbb{N}$  satisfy  $U_i = 0$ . Assume that this sequence  $(U_0, U_1, U_2, \dots)$  is  $(n+1)$ -supported. Assume that  $\partial'_g \left( \sum_{i \in \mathbb{N}} U_i q^i \right) = 0$ . Our goal now is to prove that every positive integer  $i$  satisfies  $U_i = 0$ . Once this is shown, it will follow that (17) holds for  $N = n+1$ , and so the induction step will be complete.

It is easy to show (using  $g(q) = 1$ ) that

$$\partial'_g (q^i) = iq^{i-1} \quad \text{for every } i \in \mathbb{N} \quad (19)$$

(where  $iq^{i-1}$  is to be understood as 0 when  $i = 0$ ). Hence, every  $i \in \mathbb{N}$  satisfies

$$\begin{aligned} \partial'_g (U_i q^i) &= \underbrace{\partial'_g (U_i)}_{=0} q^i + U_i \partial'_g (q^i) && \text{(by Proposition 7.3 (b))} \\ &\quad \text{(since } U_i \in (M + \overline{g}')^* \text{ and } \partial'_g ((M + \overline{g}')^*) = 0) \\ &= U_i \underbrace{\partial'_g (q^i)}_{=iq^{i-1}} = U_i \cdot iq^{i-1}. \end{aligned}$$

Now,

$$\begin{aligned} \partial'_g \left( \sum_{i \in \mathbb{N}} U_i q^i \right) &= \sum_{i \in \mathbb{N}} \underbrace{\partial'_g (U_i q^i)}_{=U_i \cdot iq^{i-1}} = \sum_{i \in \mathbb{N}} U_i \cdot iq^{i-1} = U_0 \cdot \underbrace{0q^{0-1}}_{=0} + \sum_{\substack{i \in \mathbb{N}; \\ i \text{ is positive}}} U_i \cdot iq^{i-1} \\ &= \sum_{\substack{i \in \mathbb{N}; \\ i \text{ is positive}}} U_i \cdot iq^{i-1} = \sum_{i \in \mathbb{N}} U_{i+1} \cdot (i+1) q^i = \sum_{i \in \mathbb{N}} (i+1) U_{i+1} q^i. \end{aligned}$$

Hence,  $\sum_{i \in \mathbb{N}} (i+1) U_{i+1} q^i = \partial'_g \left( \sum_{i \in \mathbb{N}} U_i q^i \right) = 0$ , so that  $\partial'_g \left( \underbrace{\sum_{i \in \mathbb{N}} (i+1) U_{i+1} q^i}_{=0} \right) = 0$ .

0.

But  $(1U_1, 2U_2, 3U_3, \dots)$  is a sequence of elements of  $(M + \overline{g}')^*$  (because  $U_1, U_2, U_3, \dots$  are elements of  $(M + \overline{g}')^*$ ), and is  $n$ -supported (since the sequence  $(U_0, U_1, U_2, \dots)$  is  $(n+1)$ -supported). Hence, we can apply Lemma 7.14 (b) to  $(1U_1, 2U_2, 3U_3, \dots)$  instead of  $(U_0, U_1, U_2, \dots)$  (because of (18)). As a result, we conclude that every positive integer  $i$  satisfies  $(i+1) U_{i+1} = 0$ . Therefore, every positive integer  $i$  satisfies  $U_{i+1} = 0$  (since the additive group  $T(L)$  is torsionfree). In other words, every integer  $i > 2$  satisfies  $U_i = 0$ .

But now, recall that  $\sum_{i \in \mathbb{N}} (i+1) U_{i+1} q^i = 0$ . Hence,

$$\begin{aligned}
0 &= \sum_{i \in \mathbb{N}} (i+1) U_{i+1} q^i = \underbrace{(0+1)}_{=1} \underbrace{U_{0+1}}_{=U_1} \underbrace{q^0}_{=1} + \sum_{\substack{i \in \mathbb{N}; \\ i \text{ is positive}}} (i+1) \underbrace{U_{i+1}}_{=0 \text{ (since } i \text{ is positive)}} q^i \\
&= U_1 + \underbrace{\sum_{\substack{i \in \mathbb{N}; \\ i \text{ is positive}}} (i+1) 0 q^i}_{=0} = U_1.
\end{aligned}$$

Hence,  $U_1 = 0$ . This (combined with the fact that every integer  $i > 2$  satisfies  $U_i = 0$ ) shows that every positive integer  $i$  satisfies  $U_i = 0$ . Thus, (17) is proven for  $N = n + 1$ . The induction step will be complete.

We have now proven (17) by induction. Hence, Lemma 7.14 (b) is proven.  $\square$

We can finally state our characteristic-zero analogue of Theorem 6.6:

**Theorem 7.15.** Assume that the  $\mathbf{k}$ -module  $L$  is free. Assume that the additive group  $\mathbf{k}$  is torsionfree. Then,  $(\bar{\mathfrak{g}}')^* = \text{Ker}(\mathfrak{t}')$ .

The proof of this theorem is similar to that of Theorem 6.6, but differs just enough that we show it in detail.

We notice that the requirement in Theorem 7.15 that the additive group  $\mathbf{k}$  is torsionfree is satisfied whenever  $\mathbf{k}$  is a commutative  $\mathbb{Q}$ -algebra, but also in cases such as  $\mathbf{k} = \mathbb{Z}$ . So Theorem 7.15 is actually a fairly general result.

*Proof of Theorem 7.15.* Proposition 7.11 shows that  $(\bar{\mathfrak{g}}')^* \subseteq \text{Ker}(\mathfrak{t}')$ . We thus only need to verify that  $\text{Ker}(\mathfrak{t}') \subseteq (\bar{\mathfrak{g}}')^*$ . This means proving that every  $U \in \text{Ker}(\mathfrak{t}')$  satisfies  $U \in (\bar{\mathfrak{g}}')^*$ . So let us fix  $U \in \text{Ker}(\mathfrak{t}')$ .

The  $\mathbf{k}$ -module  $L$  is free. Hence, the  $\mathbf{k}$ -module  $T(L)$  is free as well. Therefore, the additive group  $T(L)$  is a direct sum of many copies of the additive group  $\mathbf{k}$ . Thus, the additive group  $T(L)$  is torsionfree (because the additive group  $\mathbf{k}$  is torsionfree).

We know that the  $\mathbf{k}$ -module  $L$  is free; it thus has a basis. Since the tensor  $U \in T(L)$  can be constructed using only finitely many elements of this basis, we can thus WLOG assume that the basis of  $L$  is finite. Let us assume this, and let us denote this basis by  $(e_1, e_2, \dots, e_n)$ .

For every  $i \in \{1, 2, \dots, n\}$ , let  $e_i^* : L \rightarrow \mathbf{k}$  be the  $\mathbf{k}$ -linear map which sends  $e_i$  to 1 and sends every other  $e_j$  to 0.

For every  $k \in \{0, 1, \dots, n\}$ , we let  $M_k$  denote the  $\mathbf{k}$ -submodule of  $L$  spanned by  $e_1, e_2, \dots, e_k$ . Thus,  $M_0 = 0$  and  $M_n = L$ . Clearly, every  $k \in \{1, 2, \dots, n\}$  satisfies

$$M_k = M_{k-1} + \mathbf{k}e_k. \quad (20)$$

For every  $k \in \{0, 1, \dots, n\}$ , we set  $\mathfrak{h}_k = M_k + \bar{\mathfrak{g}}'$  and  $H_k = \mathfrak{h}_k^*$ .

Notice that  $\mathfrak{h}_n = M_n + \bar{\mathfrak{g}}' \supseteq M_n = L$  and thus  $H_n = \mathfrak{h}_n^* \supseteq L^* = T(L)$ . Hence,  $H_n = T(L)$ . Now,  $U \in T(L) = H_n$ .

On the other hand, the definition of  $\mathfrak{h}_0$  yields  $\mathfrak{h}_0 = \underbrace{M_0}_{=0} + \bar{\mathfrak{g}}' = \bar{\mathfrak{g}}'$  and thus

$$H_0 = \mathfrak{h}_0^* = (\bar{\mathfrak{g}}')^*.$$

We shall now prove that every  $k \in \{1, 2, \dots, n\}$  satisfies the following implication:

$$\text{if } U \in H_k, \text{ then } U \in H_{k-1}. \quad (21)$$

Once this is proven, we will be able to argue that  $U \in H_n$  (as we know), thus  $U \in H_{n-1}$  (by (21)), thus  $U \in H_{n-2}$  (by (21) again), and so on – until we finally arrive at  $U \in H_0$ . Since  $H_0 = (\bar{\mathfrak{g}}')^*$ , this rewrites as  $U \in (\bar{\mathfrak{g}}')^*$ , and thus we are done.

Therefore, it only remains to prove (21). So let us fix  $k \in \{1, 2, \dots, n\}$ , and assume that  $U \in H_k$ . We now need to show that  $U \in H_{k-1}$ .

We have

$$\begin{aligned} [e_k, \mathfrak{h}_{k-1}] &= \left[ \underbrace{e_k}_{\in L}, \underbrace{M_{k-1}}_{\subseteq L} \right] + \left[ \underbrace{e_k}_{\in L}, \bar{\mathfrak{g}}' \right] \quad (\text{since } \mathfrak{h}_{k-1} = M_{k-1} + \bar{\mathfrak{g}}') \\ &\subseteq \underbrace{[L, L]}_{=L'_2 \subseteq L'_2 + L'_3 + L'_4 + \dots = \bar{\mathfrak{g}}'} + \underbrace{[L, \bar{\mathfrak{g}}']}_{\subseteq \bar{\mathfrak{g}}'} \subseteq \bar{\mathfrak{g}}' + \bar{\mathfrak{g}}' = \bar{\mathfrak{g}}' \\ &\quad (\text{by Proposition 7.9}) \\ &\subseteq M_{k-1} + \bar{\mathfrak{g}}' = \mathfrak{h}_{k-1} \subseteq \mathfrak{h}_{k-1}^*. \end{aligned}$$

Thus, Lemma 7.12 (applied to  $u = e_k$  and  $S = \mathfrak{h}_{k-1}$ ) yields that  $\sum_{j \in \mathbb{N}} \mathfrak{h}_{k-1}^* e_k^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ . In other words,  $\sum_{j \in \mathbb{N}} H_{k-1} e_k^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$  (since  $H_{k-1} = \mathfrak{h}_{k-1}^*$ ). This  $\mathbf{k}$ -subalgebra contains  $\mathfrak{h}_k$  as a subset<sup>17</sup>, and thus we have  $H_k \subseteq \sum_{j \in \mathbb{N}} H_{k-1} e_k^j$ <sup>18</sup>. (Actually,  $H_k = \sum_{j \in \mathbb{N}} H_{k-1} e_k^j$ , but we don't need this.)

<sup>17</sup>Proof. We have  $\mathfrak{h}_k = M_k + \bar{\mathfrak{g}}'$  and similarly  $\mathfrak{h}_{k-1} = M_{k-1} + \bar{\mathfrak{g}}'$ . Thus,

$$\begin{aligned} \mathfrak{h}_k &= \underbrace{M_k}_{=M_{k-1} + \mathbf{k}e_k \text{ (by (20))}} + \bar{\mathfrak{g}}' = M_{k-1} + \mathbf{k}e_k + \bar{\mathfrak{g}}' = \underbrace{M_{k-1} + \bar{\mathfrak{g}}'}_{= \mathfrak{h}_{k-1} \subseteq \mathfrak{h}_{k-1}^* = H_{k-1} \text{ (since } H_{k-1} \text{ was defined as } \mathfrak{h}_{k-1}^*)} + \underbrace{\mathbf{k}}_{\subseteq H_{k-1}} e_k \\ &\subseteq H_{k-1} + H_{k-1} e_k \subseteq \sum_{j \in \mathbb{N}} H_{k-1} e_k^j \end{aligned}$$

(since  $H_{k-1}$  and  $H_{k-1} e_k$  are the first two addends of the sum  $\sum_{j \in \mathbb{N}} H_{k-1} e_k^j$ ), qed.

<sup>18</sup>Proof. The  $\mathbf{k}$ -subalgebra  $\sum_{j \in \mathbb{N}} H_{k-1} e_k^j$  of  $T(L)$  contains  $\mathfrak{h}_k$  as a subset. Hence, it also con-



Now,  $U \in H_k \subseteq \sum_{j \in \mathbb{N}} H_{k-1} e_k^j = \sum_{i \in \mathbb{N}} H_{k-1} e_k^i$ . Therefore, there exists a sequence  $(U_0, U_1, U_2, \dots)$  of elements of  $H_{k-1}$  such that all but finitely many  $i \in \mathbb{N}$  satisfy  $U_i = 0$  and such that we have  $U = \sum_{i \in \mathbb{N}} U_i e_k^i$ . Consider this sequence  $(U_0, U_1, U_2, \dots)$ . We have

$$\partial'_{e_k^*} \left( \underbrace{\sum_{i \in \mathbb{N}} U_i e_k^i}_{=U} \right) = \partial'_{e_k^*} \left( \underbrace{U}_{\in \text{Ker}(\mathbf{t}')} \right) \in \partial'_{e_k^*} (\text{Ker}(\mathbf{t}')) = 0$$

(by Proposition 7.4 (a), applied to  $g = e_k^*$ ), so that  $\partial'_{e_k^*} \left( \sum_{i \in \mathbb{N}} U_i e_k^i \right) = 0$ .

The entries  $U_0, U_1, U_2, \dots$  of the sequence  $(U_0, U_1, U_2, \dots)$  belong to  $H_{k-1} = \mathfrak{h}_{k-1}^* = (M_{k-1} + \overline{\mathfrak{g}}')^*$  (since  $\mathfrak{h}_{k-1} = M_{k-1} + \overline{\mathfrak{g}}'$ ). We can thus apply Lemma 7.14 (b) to  $M = M_{k-1}$ ,  $g = e_k^*$  and  $q = e_k$  (since  $e_k^*(M_{k-1}) = 0$  (because  $M_{k-1}$  is spanned by  $e_1, e_2, \dots, e_{k-1}$ ) and  $e_k^*(e_k) = 1$  and  $\partial'_{e_k^*} \left( \sum_{i \in \mathbb{N}} U_i e_k^i \right) = 0$ ). As a result, we see that every positive integer  $i$  satisfies  $U_i = 0$ . Thus,

$$U = \sum_{i \in \mathbb{N}} U_i e_k^i = U_0 \underbrace{e_k^0}_{=1} + \sum_{\substack{i \in \mathbb{N}; \\ i \text{ is positive}}} \underbrace{U_i}_{=0 \text{ (since } i \text{ is positive)}} e_k^i = U_0 + \underbrace{\sum_{\substack{i \in \mathbb{N}; \\ i \text{ is positive}}} 0 e_k^i}_{=0} = U_0 \in H_{k-1}.$$

This completes the proof of (21). As we already mentioned, this finishes the proof of Theorem 7.15.  $\square$

## 8. The even analogue in positive characteristic

We now come to the question of determining  $\text{Ker}(\mathbf{t}')$  when the ground ring  $\mathbf{k}$  is an  $\mathbb{F}_p$ -algebra for a prime number  $p$ . In this case, as we will see, a  $\mathbf{k}$ -submodule similar to the  $P$  of Definition 5.3 will become relevant once again.

**Convention 8.1.** For this whole section, we fix a prime number  $p$ , and we assume that  $\mathbf{k}$  is a commutative  $\mathbb{F}_p$ -algebra.

This assumption yields that  $p = 0$  in  $\mathbf{k}$ . Let us immediately put this to use by stating an analogue of Proposition 4.4 (e) (which, as we recall, had no analogue in the case of arbitrary  $\mathbf{k}$ ):

---

tains  $\mathfrak{h}_k^*$  as a subset (since  $\mathfrak{h}_k^*$  is the  $\mathbf{k}$ -subalgebra of  $T(L)$  generated by  $\mathfrak{h}_k$ ). In other words,  $\sum_{j \in \mathbb{N}} H_{k-1} e_k^j \supseteq \mathfrak{h}_k^* = H_k$ , qed.

■ **Proposition 8.2.** We have  $x^p \in \text{Ker}(\mathbf{t}')$  for each  $x \in L$ .

*Proof of Proposition 8.2.* Let  $x \in L$ . It is easy to see that  $\mathbf{t}'(x^i) = ix^i$  for every positive integer  $i$ . Applying this to  $i = p$ , we obtain  $\mathbf{t}'(x^p) = px^p = 0$  (since  $p = 0$  in  $\mathbf{k}$ ). Thus,  $x^p \in \text{Ker}(\mathbf{t}')$ . This proves Proposition 8.2.  $\square$

■ **Definition 8.3.** Let  $P_p$  denote the  $\mathbf{k}$ -submodule of  $L^{\otimes p}$  spanned by elements of the form  $\underbrace{x \otimes x \otimes \cdots \otimes x}_{p \text{ times}}$  with  $x \in L$ . Notice that  $\underbrace{x \otimes x \otimes \cdots \otimes x}_{p \text{ times}} = x^p$  in the  $\mathbf{k}$ -algebra  $T(L)$  for every  $x \in L$ .

Of course, when  $p = 2$ , we have  $P_p = P_2 = P$ .

■ **Definition 8.4.** Let  $\mathfrak{h}'_p = \overline{\mathfrak{g}}' + P_p$ .

Recall that Proposition 7.9 was an analogue of part of Proposition 5.6. Now that we have  $\mathbf{k}$ -submodules  $P_p$  and  $\mathfrak{h}'_p$  similar to our formerly defined  $P$  and  $\mathfrak{h}$ , we can state an analogue of the remainder of that proposition:

■ **Proposition 8.5.** We have  $[L, P_p] \subseteq \overline{\mathfrak{g}}'$  and  $[L, \mathfrak{h}'_p] \subseteq \overline{\mathfrak{g}}' \subseteq \mathfrak{h}'_p$ .

*Proof of Proposition 8.5.* Let us first make some general observations on commutators in  $\mathbf{k}$ -algebras.

Let  $A$  be any  $\mathbf{k}$ -algebra. Clearly, there is a commutator  $[\cdot, \cdot]$  on  $A$ , defined by  $[U, V] = UV - VU$  for all  $U \in A$  and  $V \in A$ .

For every  $a \in A$ , we define a  $\mathbf{k}$ -linear map  $\text{ad}_a : A \rightarrow A$  by setting

$$\text{ad}_a(c) = [a, c] \quad \text{for all } c \in A.$$

Then, every  $a \in A$  satisfies

$$\text{ad}_{a^p} = (\text{ad}_a)^p \tag{22}$$

<sup>19</sup>.

---

<sup>19</sup>*Proof of (22):* We first notice that  $(-1)^p = -1$  in  $\mathbf{k}$ . (This is because Fermat's little theorem yields  $(-1)^p \equiv -1 \pmod{p}$ .)

For every  $a \in A$ , we define a  $\mathbf{k}$ -linear map  $\mathcal{L}_a : A \rightarrow A$  by setting

$$\mathcal{L}_a(c) = ac \quad \text{for all } c \in A.$$

For every  $a \in A$ , we define a  $\mathbf{k}$ -linear map  $\mathcal{R}_a : A \rightarrow A$  by setting

$$\mathcal{R}_a(c) = ca \quad \text{for all } c \in A.$$

It is easy to see that  $\mathcal{L}_0 = 0$  and  $\mathcal{L}_1 = \text{id}$ , and that any  $a \in A$  and  $b \in A$  satisfy  $\mathcal{L}_{ab} = \mathcal{L}_a \circ \mathcal{L}_b$  and  $\mathcal{L}_{a+b} = \mathcal{L}_a + \mathcal{L}_b$ . Hence, the map

$$A \rightarrow \text{End } A, \quad a \mapsto \mathcal{L}_a$$

Now, let  $A = T(L)$ . Fix  $x \in L$ . Then,

$$\begin{aligned} \text{ad}_x(\bar{\mathfrak{g}}') &= \left[ \underbrace{x}_{\in L}, \bar{\mathfrak{g}}' \right] && (\text{since } \text{ad}_x(c) = [x, c] \text{ for all } c \in \bar{\mathfrak{g}}') \\ &\subseteq [L, \bar{\mathfrak{g}}'] \subseteq \bar{\mathfrak{g}}' && (\text{by Proposition 7.9}). \end{aligned}$$

Thus, the map  $\text{ad}_x : T(L) \rightarrow T(L)$  restricts to a map  $\text{ad}_x : \bar{\mathfrak{g}}' \rightarrow \bar{\mathfrak{g}}'$ . Consequently,

$$(\text{ad}_x)^n(\bar{\mathfrak{g}}') \subseteq \bar{\mathfrak{g}}' \quad \text{for every } n \in \mathbb{N}. \quad (23)$$

---

is a  $\mathbf{k}$ -algebra homomorphism. Consequently,  $\mathcal{L}_{a^p} = (\mathcal{L}_a)^p$  for every  $a \in A$ . Similarly,  $\mathcal{R}_{a^p} = (\mathcal{R}_a)^p$  for every  $a \in A$ .

Now, fix  $a \in A$ . Then, the maps  $\mathcal{L}_a$  and  $\mathcal{R}_a$  commute. (In fact, more generally, the maps  $\mathcal{L}_a$  and  $\mathcal{R}_b$  commute for every  $b \in A$ . This is straightforward to check.) Moreover,  $\text{ad}_a = \mathcal{L}_a - \mathcal{R}_a$  (since every  $c \in A$  satisfies  $\text{ad}_a(c) = [a, c] = \underbrace{ac}_{=\mathcal{L}_a(c)} - \underbrace{ca}_{=\mathcal{R}_a(c)} = \mathcal{L}_a(c) - \mathcal{R}_a(c) =$

$(\mathcal{L}_a - \mathcal{R}_a)(c)$ ). The same argument (applied to  $a^p$  instead of  $a$ ) shows that  $\text{ad}_{a^p} = \mathcal{L}_{a^p} - \mathcal{R}_{a^p}$ . Now, the maps  $\mathcal{L}_a$  and  $\mathcal{R}_a$  commute; thus, we can apply the binomial formula to them. We thus obtain

$$\begin{aligned} &(\mathcal{L}_a - \mathcal{R}_a)^p \\ &= \sum_{k=0}^p \binom{p}{k} (\mathcal{L}_a)^k (-\mathcal{R}_a)^{p-k} \\ &= \underbrace{\binom{p}{0}}_{=1} \underbrace{(\mathcal{L}_a)^0}_{=1} \underbrace{(-\mathcal{R}_a)^{p-0}}_{=(-\mathcal{R}_a)^p} + \sum_{k=1}^{p-1} \underbrace{\binom{p}{k}}_{\substack{=0 \text{ in } \mathbf{k} \\ (\text{since } p \mid \binom{p}{k}) \\ (\text{because } p \text{ is prime and } 0 < k < p)}} (\mathcal{L}_a)^k (-\mathcal{R}_a)^{p-k} + \underbrace{\binom{p}{p}}_{=1} (\mathcal{L}_a)^p \underbrace{(-\mathcal{R}_a)^{p-p}}_{=(-\mathcal{R}_a)^0=1} \\ &= \underbrace{(-\mathcal{R}_a)^p}_{\substack{=(-1)^p(\mathcal{R}_a)^p = -(\mathcal{R}_a)^p \\ (\text{since } (-1)^p = -1 \text{ in } \mathbf{k})}} + \underbrace{\sum_{k=1}^{p-1} 0 (\mathcal{L}_a)^k (-\mathcal{R}_a)^{p-k}}_{=0} + (\mathcal{L}_a)^p \\ &= -\underbrace{(\mathcal{R}_a)^p}_{=\mathcal{R}_{a^p}} + \underbrace{(\mathcal{L}_a)^p}_{=\mathcal{L}_{a^p}} = -\mathcal{R}_{a^p} + \mathcal{L}_{a^p} = \mathcal{L}_{a^p} - \mathcal{R}_{a^p} = \text{ad}_{a^p}. \end{aligned}$$

Since  $\text{ad}_a = \mathcal{L}_a - \mathcal{R}_a$ , this rewrites as  $(\text{ad}_a)^p = \text{ad}_{a^p}$ . This proves (22).

Now, if  $m$  is a positive integer, then

$$\begin{aligned}
\underbrace{(\text{ad}_x)^m}_{=(\text{ad}_x)^{m-1} \circ \text{ad}_x} (L) &= \left( (\text{ad}_x)^{m-1} \circ \text{ad}_x \right) (L) = (\text{ad}_x)^{m-1} \underbrace{(\text{ad}_x (L))}_{=[x, L]} \\
&= (\text{ad}_x)^{m-1} \left( \left[ \underbrace{x}_{\in L}, L \right] \right) \in (\text{ad}_x)^{m-1} \left( \begin{array}{c} [L, L] \\ = L'_2 \\ \subseteq L'_2 + L'_3 + L'_4 + \dots \\ = \bar{\mathfrak{g}}' \end{array} \right) \\
&\in (\text{ad}_x)^{m-1} (\bar{\mathfrak{g}}') \subseteq \bar{\mathfrak{g}}' \quad (\text{by (23), applied to } n = m - 1). \tag{24}
\end{aligned}$$

Hence, every  $U \in L$  satisfies

$$\begin{aligned}
[U, x^p] &= - \underbrace{[x^p, U]}_{=\text{ad}_{x^p}(U)} = - \underbrace{\text{ad}_{x^p}}_{=(\text{ad}_x)^p \text{ (by (22))}} \left( \underbrace{U}_{\in L} \right) \in - \underbrace{(\text{ad}_x)^p (L)}_{\subseteq \bar{\mathfrak{g}}' \text{ (by (24))}} \subseteq -\bar{\mathfrak{g}}' \subseteq \bar{\mathfrak{g}}'.
\end{aligned}$$

Let us now forget that we fixed  $x$  and  $U$ . We thus have shown that  $[U, x^p] \in \bar{\mathfrak{g}}'$  for every  $U \in L$  and every  $x \in L$ . This shows that  $[L, P_p] \subseteq \bar{\mathfrak{g}}'$  (because the  $\mathbf{k}$ -module  $P_p$  is spanned by elements of the form  $x^p$  with  $x \in L$ ).

It remains to prove that  $[L, \mathfrak{h}'_p] \subseteq \bar{\mathfrak{g}}' \subseteq \mathfrak{h}'_p$ . But this is easy: We have  $\mathfrak{h}'_p = \bar{\mathfrak{g}}' + P_p$ , and thus

$$\begin{aligned}
[L, \mathfrak{h}'_p] &= \underbrace{[L, \bar{\mathfrak{g}}']}_{\subseteq \bar{\mathfrak{g}}' \text{ (by Proposition 7.9)}} + \underbrace{[L, P_p]}_{\subseteq \bar{\mathfrak{g}}'} \subseteq \bar{\mathfrak{g}}' + \bar{\mathfrak{g}}' \subseteq \bar{\mathfrak{g}}' \subseteq \bar{\mathfrak{g}}' + P_p = \mathfrak{h}'_p.
\end{aligned}$$

This proves Proposition 8.5. □

Next, we state an analogue of the parts of Proposition 5.7 not covered by Proposition 7.10:

**Proposition 8.6. (a)** We have  $[\mathfrak{h}'_p, \mathfrak{h}'_p] \subseteq \bar{\mathfrak{g}}' \subseteq \mathfrak{h}'_p$ .

**(b)** The two  $\mathbf{k}$ -submodules  $\mathfrak{h}'_p$  and  $\mathfrak{h}'_p + L$  of  $T(L)$  are invariant under the commutator  $[\cdot, \cdot]$ . (In other words, they are Lie subalgebras of  $T(L)$  (with the commutator  $[\cdot, \cdot]$  as the Lie bracket).)

*Proof of Proposition 8.6. (a)* First, we notice that

$$\left[ P_p, \sum_{i \geq 1} L'_i \right] \subseteq \bar{\mathfrak{g}}'. \tag{25}$$

Next, we notice that any two positive integers  $i$  and  $j$  satisfy

$$[L'_i, L'_j] \subseteq L'_{i+j}. \quad (26)$$

Now,

$$\begin{aligned} \left[ \sum_{i \geq 1} L'_i, \sum_{i \geq 1} L'_i \right] &= \left[ \sum_{i \geq 1} L'_i, \sum_{j \geq 1} L'_j \right] = \sum_{i \geq 1} \sum_{j \geq 1} \underbrace{[L'_i, L'_j]}_{\substack{\subseteq L'_{i+j} \\ \text{(by (26))}}} \subseteq \sum_{i \geq 1} \sum_{j \geq 1} L'_{i+j} \\ &\subseteq \sum_{k \geq 2} L'_k \quad (\text{since } i + j \geq 2 \text{ for any } i \geq 1 \text{ and } j \geq 1) \\ &= L'_2 + L'_3 + L'_4 + \cdots = \bar{\mathfrak{g}}'. \end{aligned}$$

Recall now that  $\bar{\mathfrak{g}}' = L'_2 + L'_3 + L'_4 + \cdots = \sum_{i \geq 2} L'_i \subseteq \sum_{i \geq 1} L'_i$ . Thus,

$$\left[ \bar{\mathfrak{g}}', \sum_{i \geq 1} L'_i \right] \subseteq \left[ \sum_{i \geq 1} L'_i, \sum_{i \geq 1} L'_i \right] \subseteq \bar{\mathfrak{g}}'.$$

---

<sup>20</sup>*Proof of (25):* It is clearly enough to show that  $[P_p, L'_i] \subseteq \bar{\mathfrak{g}}'$  for all positive integers  $i$ . So let us do this. Let  $i$  be a positive integer. We need to show that  $[P_p, L'_i] \subseteq \bar{\mathfrak{g}}'$ . In other words, we need to show that  $[x^p, L'_i] \subseteq \bar{\mathfrak{g}}'$  for every  $x \in L$  (because the  $\mathbf{k}$ -module  $P_p$  is spanned by elements of the form  $x^p$  with  $x \in L$ ). So let us fix  $x \in L$ . We need to prove  $[x^p, L'_i] \subseteq \bar{\mathfrak{g}}'$ .

We shall use the notations introduced in the proof of Proposition 8.5. Let  $U \in L'_i$ . Then, the definition of  $\text{ad}_x$  yields

$$\begin{aligned} \text{ad}_x(U) &= \left[ \underbrace{x}_{\in L}, \underbrace{U}_{\in L'_i} \right] \in [L, L'_i] = L'_{i+1} \subseteq L'_2 + L'_3 + L'_4 + \cdots \quad (\text{since } i + 1 \geq 2) \\ &= \bar{\mathfrak{g}}'. \end{aligned}$$

But the definition of  $\text{ad}_{x^p}$  yields  $\text{ad}_{x^p}(U) = [x^p, U]$ , so that

$$\begin{aligned} [x^p, U] &= \underbrace{\text{ad}_{x^p}}_{\substack{= (\text{ad}_x)^p \\ \text{(by (22))}}}(U) = \underbrace{(\text{ad}_x)^p}_{= (\text{ad}_x)^{p-1} \circ \text{ad}_x}(U) = ((\text{ad}_x)^{p-1} \circ \text{ad}_x)(U) \\ &= (\text{ad}_x)^{p-1} \left( \underbrace{\text{ad}_x(U)}_{\in \bar{\mathfrak{g}}} \right) \in (\text{ad}_x)^{p-1}(\bar{\mathfrak{g}}) \subseteq \bar{\mathfrak{g}}' \quad (\text{by (23)}). \end{aligned}$$

Let us now forget that we fixed  $U$ . We thus have shown that  $[x^p, U] \in \bar{\mathfrak{g}}'$  for every  $U \in L'_i$ . In other words,  $[x^p, L'_i] \subseteq \bar{\mathfrak{g}}'$ . This completes our proof of (25).

<sup>21</sup>The proof of (26) is analogous to the proof of (7) given earlier in this note.

Since  $\mathfrak{h}'_p = \overline{\mathfrak{g}}' + P_p$ , we have

$$\left[ \mathfrak{h}'_p, \sum_{i \geq 1} L'_i \right] = \underbrace{\left[ \overline{\mathfrak{g}}', \sum_{i \geq 1} L'_i \right]}_{\subseteq \overline{\mathfrak{g}}'} + \underbrace{\left[ P_p, \sum_{i \geq 1} L'_i \right]}_{\substack{\subseteq \overline{\mathfrak{g}}' \\ \text{(by (25))}}} \subseteq \overline{\mathfrak{g}}' + \overline{\mathfrak{g}}' = \overline{\mathfrak{g}}'.$$

Since  $\overline{\mathfrak{g}}' \subseteq \sum_{i \geq 1} L'_i$ , we now have

$$\left[ \mathfrak{h}'_p, \overline{\mathfrak{g}}' \right] \subseteq \left[ \mathfrak{h}'_p, \sum_{i \geq 1} L'_i \right] \subseteq \overline{\mathfrak{g}}'. \quad (27)$$

But we also have  $L = L'_1 \subseteq \sum_{i \geq 1} L'_i$  and thus

$$\left[ \mathfrak{h}'_p, L \right] \subseteq \left[ \mathfrak{h}'_p, \sum_{i \geq 1} L'_i \right] \subseteq \overline{\mathfrak{g}}'. \quad (28)$$

From this, we easily obtain

$$\left[ \mathfrak{h}'_p, P_p \right] \subseteq \overline{\mathfrak{g}}'$$

22.

---

<sup>22</sup>*Proof.* It is clearly enough to show that  $\left[ \mathfrak{h}'_p, x^p \right] \subseteq \overline{\mathfrak{g}}'$  for every  $x \in L$  (since the  $\mathbf{k}$ -module  $P_p$  is spanned by elements of the form  $x^p$  for  $x \in L$ ). So let  $x \in L$ .

We shall use the notations introduced in the proof of Proposition 8.5. Let  $U \in \mathfrak{h}'_p$ . Then, the definition of  $\text{ad}_x$  yields

$$\text{ad}_x(U) = \left[ \underbrace{x}_{\in L}, \underbrace{U}_{\in \mathfrak{h}'_p} \right] \in \left[ L, \mathfrak{h}'_p \right] = \left[ \mathfrak{h}'_p, L \right] \subseteq \overline{\mathfrak{g}}' \quad (\text{by (28)}).$$

But the definition of  $\text{ad}_{x^p}$  yields  $\text{ad}_{x^p}(U) = [x^p, U]$ , so that

$$\begin{aligned} [x^p, U] &= \underbrace{\text{ad}_{x^p}}_{\substack{= (\text{ad}_x)^p \\ \text{(by (22))}}}(U) = \underbrace{(\text{ad}_x)^p}_{= (\text{ad}_x)^{p-1} \circ \text{ad}_x}(U) = ((\text{ad}_x)^{p-1} \circ \text{ad}_x)(U) \\ &= (\text{ad}_x)^{p-1} \left( \underbrace{\text{ad}_x(U)}_{\in \overline{\mathfrak{g}}'} \right) \in (\text{ad}_x)^{p-1}(\overline{\mathfrak{g}}') \subseteq \overline{\mathfrak{g}}' \quad (\text{by (23)}). \end{aligned}$$

Hence,  $[U, x^p] = -\underbrace{[x^p, U]}_{\in \overline{\mathfrak{g}}'} \in \overline{\mathfrak{g}}'.$

Let us now forget that we fixed  $U$ . We thus have shown that  $[U, x^p] \in \overline{\mathfrak{g}}'$  for every  $U \in \mathfrak{h}'_p$ . In other words,  $\left[ \mathfrak{h}'_p, x^p \right] \subseteq \overline{\mathfrak{g}}'$ , qed.

Now, using  $\mathfrak{h}'_p = \bar{\mathfrak{g}}' + P_p$  again, we obtain

$$[\mathfrak{h}'_p, \mathfrak{h}'_p] = \underbrace{[\mathfrak{h}'_p, \bar{\mathfrak{g}}']}_{\substack{\subseteq \bar{\mathfrak{g}}' \\ \text{(by (27))}}} + \underbrace{[\mathfrak{h}'_p, P_p]}_{\subseteq \bar{\mathfrak{g}}'} \subseteq \bar{\mathfrak{g}}' + \bar{\mathfrak{g}}' = \bar{\mathfrak{g}}'.$$

This proves Proposition 8.6 (a).

(b) We need to show that  $[\mathfrak{h}'_p, \mathfrak{h}'_p] \subseteq \mathfrak{h}'_p$  and  $[\mathfrak{h}'_p + L, \mathfrak{h}'_p + L] \subseteq \mathfrak{h}'_p + L$ .

The relation  $[\mathfrak{h}'_p, \mathfrak{h}'_p] \subseteq \mathfrak{h}'_p$  follows from  $[\mathfrak{h}'_p, \mathfrak{h}'_p] \subseteq \bar{\mathfrak{g}}' \subseteq \mathfrak{h}'_p$ .

We have

$$\begin{aligned} [\mathfrak{h}'_p + L, \mathfrak{h}'_p + L] &= \underbrace{[\mathfrak{h}'_p, \mathfrak{h}'_p]}_{\subseteq \bar{\mathfrak{g}}'} + \underbrace{[\mathfrak{h}'_p, L]}_{\substack{\subseteq \bar{\mathfrak{g}}' \\ \text{(by (28))}}} + \underbrace{[L, \mathfrak{h}'_p]}_{\substack{=[\mathfrak{h}'_p, L] \subseteq \bar{\mathfrak{g}}' \\ \text{(by (28))}}} + \underbrace{[L, L]}_{=L'_2 \subseteq L'_2 + L'_3 + L'_4 + \dots = \bar{\mathfrak{g}}'} \\ &\subseteq \bar{\mathfrak{g}}' + \bar{\mathfrak{g}}' + \bar{\mathfrak{g}}' + \bar{\mathfrak{g}}' = \bar{\mathfrak{g}}' \subseteq \mathfrak{h}'_p \subseteq \mathfrak{h}'_p + L. \end{aligned}$$

Proposition 8.6 (b) is thus shown.  $\square$

Next, we state an analogue of Proposition 6.2 which is stronger than Proposition 7.11 (of course under the assumption that  $\mathbf{k}$  is an  $\mathbb{F}_p$ -algebra):

**Proposition 8.7.** We have  $(\mathfrak{h}'_p)^* \subseteq \text{Ker}(\mathfrak{t}')$ .

*Proof of Proposition 8.7.* We have  $P_p \subseteq \text{Ker}(\mathfrak{t}')$  due to Proposition 8.2. Also,  $L'_2 = [L, L] \subseteq \text{Ker}(\mathfrak{t}')$  by Proposition 7.6 (c). Using this and Proposition 7.6 (d), we can show that  $L'_i \subseteq \text{Ker}(\mathfrak{t}')$  for each  $i \geq 2$  (by induction over  $i$ ). Thus,  $\bar{\mathfrak{g}}' \subseteq \text{Ker}(\mathfrak{t}')$  (since  $\bar{\mathfrak{g}}' = L'_2 + L'_3 + L'_4 + \dots$ ). Combined with  $P_p \subseteq \text{Ker}(\mathfrak{t}')$ , this yields  $\mathfrak{h}'_p \subseteq \text{Ker}(\mathfrak{t}')$  (since  $\mathfrak{h}'_p = \bar{\mathfrak{g}}' + P_p$ ). Since  $\text{Ker}(\mathfrak{t}')$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$  (because of Proposition 7.6 (a) and Proposition 7.6 (b)), this yields that  $(\mathfrak{h}'_p)^* \subseteq \text{Ker}(\mathfrak{t}')$ . This proves Proposition 8.7.  $\square$

Next, we state some lemmas. The first lemma is an analogue of Lemma 6.3 again:

**Lemma 8.8.** Let  $u \in L$ . Let  $S$  be a  $\mathbf{k}$ -submodule of  $T(L)$  such that  $u^p \in S^*$  and such that  $[u, S] \subseteq S^*$ . Then,  $\sum_{j=0}^{p-1} S^* u^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ .

*Proof of Lemma 8.8.* Lemma 7.12 yields that  $\sum_{j \in \mathbb{N}} S^* u^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ .

We shall now show that  $\sum_{j \in \mathbb{N}} S^* u^j = \sum_{j=0}^{p-1} S^* u^j$ .

We have  $S^*u^p \subseteq S^*u^0$ <sup>23</sup>. Thus,

$$S^*u^k \subseteq \sum_{j=0}^{p-1} S^*u^j \quad \text{for every } k \in \mathbb{N}. \quad (29)$$

<sup>24</sup> Now,

$$\begin{aligned} \sum_{j \in \mathbb{N}} S^*u^j &= \sum_{k \in \mathbb{N}} \underbrace{S^*u^k}_{\substack{\subseteq \sum_{j=0}^{p-1} S^*u^j \\ \text{(by (29))}}} \subseteq \sum_{k \in \mathbb{N}} \sum_{j=0}^{p-1} S^*u^j \subseteq \sum_{j=0}^{p-1} S^*u^j. \end{aligned}$$

Combined with  $\sum_{j=0}^{p-1} S^*u^j \subseteq \sum_{j \in \mathbb{N}} S^*u^j$  (this is obvious), this yields  $\sum_{j \in \mathbb{N}} S^*u^j = \sum_{j=0}^{p-1} S^*u^j$ . Hence,  $\sum_{j=0}^{p-1} S^*u^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$  (since  $\sum_{j \in \mathbb{N}} S^*u^j$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ ). This proves Lemma 8.8.  $\square$

Next comes, again, an analogue of Lemma 6.5:

---

<sup>23</sup>*Proof.* We have  $S^* \underbrace{u^p}_{\in S^*} = S^*S^* \subseteq S^*$  (since  $S^*$  is a  $\mathbf{k}$ -algebra). But  $S^* \underbrace{u^0}_{=1} = S^*$ , so that

$S^*u^p \subseteq S^* = S^*u^0$ , qed.

<sup>24</sup>*Proof of (29):* We shall prove (29) by induction over  $k$ .

*Induction base:* We have  $S^*u^0 \subseteq \sum_{j=0}^{p-1} S^*u^j$  (since  $S^*u^0$  is an addend of the sum  $\sum_{j=0}^{p-1} S^*u^j$ ). In other words, (29) holds for  $k = 0$ . This completes the induction base.

*Induction step:* Let  $K \in \mathbb{N}$ . Assume that (29) holds for  $k = K$ . We must prove that (29) holds for  $k = K + 1$ .

We have  $S^*u^K \subseteq \sum_{j=0}^{p-1} S^*u^j$  (since (29) holds for  $k = K$ ). Hence,

$$\begin{aligned} S^* \underbrace{u^{K+1}}_{=u^K u} &= \underbrace{S^*u^K}_{\subseteq \sum_{j=0}^{p-1} S^*u^j} u \subseteq \left( \sum_{j=0}^{p-1} S^*u^j \right) u \\ &= \sum_{j=0}^{p-1} S^* \underbrace{u^j u}_{=u^{j+1}} = \sum_{j=0}^{p-1} S^*u^{j+1} = \sum_{j=1}^p S^*u^j \\ &= \sum_{j=1}^{p-1} S^*u^j + \underbrace{S^*u^p}_{\subseteq S^*u^0} \subseteq \sum_{j=1}^{p-1} S^*u^j + S^*u^0 = \sum_{j=0}^{p-1} S^*u^j. \end{aligned}$$

In other words, (29) holds for  $k = K + 1$ . This completes the induction step. Thus, (29) is proven.



**Lemma 8.9.** Let  $M$  be a  $\mathbf{k}$ -submodule of  $L$ . Let  $g \in L^*$  be such that  $g(M) = 0$ . Then:

(a) We have  $\partial'_g \left( (M + \mathfrak{h}'_p)^* \right) = 0$ .

(b) Let  $q \in L$  be such that  $g(q) = 1$ . Let  $(U_0, U_1, \dots, U_{p-1})$  be a  $p$ -tuple of elements of  $(M + \mathfrak{h}'_p)^*$ . If  $\partial'_g \left( \sum_{i=0}^{p-1} U_i q^i \right) = 0$ , then every  $i \in \{1, 2, \dots, p-1\}$  satisfies  $U_i = 0$ .

*Proof of Lemma 8.9.* The proof of Lemma 8.9 (a) is analogous to that of Lemma 6.5 (a).

The proof of Lemma 8.9 (b) is analogous to that of Lemma 7.14 (b) (with some rather obvious changes:  $\bar{g}'$  has to be replaced by  $\mathfrak{h}'_p$ ; the sequence  $(U_0, U_1, U_2, \dots)$  has to be replaced by the  $p$ -tuple  $(U_0, U_1, \dots, U_{p-1})$ ; the assumption that the additive group  $T(L)$  is torsionfree has to be replaced by the obvious observation that the integers  $1, 2, \dots, p-1$  are invertible in  $\mathbf{k}$  (since  $p = 0$  in  $\mathbf{k}$ )).  $\square$

The positive-characteristic version of Theorem 7.15 can now be stated and proven:

**Theorem 8.10.** Assume that the  $\mathbf{k}$ -module  $L$  is free. Then,  $(\mathfrak{h}'_p)^* = \text{Ker}(\mathfrak{t}')$ .

*Proof of Theorem 8.10.* The proof of Theorem 8.10 is more or less analogous to that of Theorem 7.15. (As usual, we need to make some replacements to the proof:

- We must replace every  $\bar{g}'$  by  $\mathfrak{h}'_p$  (with a few exceptions: for instance,  $L'_2 + L'_3 + L'_4 + \dots = \bar{g}'$  should become  $L'_2 + L'_3 + L'_4 + \dots = \bar{g}' \subseteq \mathfrak{h}'_p$  rather than  $L'_2 + L'_3 + L'_4 + \dots = \mathfrak{h}'_p$ ).
- The claim that the additive group  $T(L)$  is torsionfree is now wrong (but we don't need this claim).
- Instead of using Proposition 7.11, we need to use Proposition 8.7.
- Instead of using Proposition 7.9, we need to use Proposition 8.5 (specifically, the part of it that says  $[L, \mathfrak{h}'_p] \subseteq \mathfrak{h}'_p$ ).
- Every summation sign  $\sum_{j \in \mathbb{N}}$  must be replaced by  $\sum_{j=0}^{p-1}$ . Similarly, every summation sign  $\sum_{i \in \mathbb{N}}$  must be replaced by  $\sum_{i=0}^{p-1}$ .
- Instead of using Lemma 7.12, we need to use Lemma 8.8.

- Instead of using Lemma 7.14, we need to use Lemma 8.9.
  - The sequence  $(U_0, U_1, U_2, \dots)$  has to be replaced by a  $p$ -tuple  $(U_0, U_1, \dots, U_{p-1})$ .
- ) □

## 9. Further questions

Above, we have described the kernel of  $\mathbf{t}$  whenever  $L$  is a free  $\mathbf{k}$ -module (Theorem 6.6), and also the kernel of  $\mathbf{t}'$  whenever  $L$  is a free  $\mathbf{k}$ -module and  $\mathbf{k}$  is either torsionfree as an additive group (Theorem 7.15) or an  $\mathbb{F}_p$ -algebra (Theorem 8.10). It is tempting to ask whether some of these conditions can be lifted; in particular, the following generalization seems viable:

**TODO 9.1.** Can some of our results be extended from the case of  $L$  free to the case of  $L$  flat? Lazard's theorem might help deducing the latter from the former.

Other natural questions include:

**TODO 9.2.** Having found the kernels of  $\mathbf{t}$  and  $\mathbf{t}'$ , the next logical step appears to be diagonalizing these maps. This has been done for  $\mathbf{t}'$  by Amy Pang [Pang15, Theorem 5.1] under the assumption that  $\mathbf{k}$  is a characteristic-0 field. Pang works with a basis, but in a basis-free language her result (or, rather, the particular case of it relevant to us) says that (when  $\mathbf{k}$  is a  $\mathbb{Q}$ -algebra) we have  $T(L) = T(L)^{\text{sym}} \cdot \text{Ker}(\mathbf{t}')$ , where  $T(L)^{\text{sym}}$  is the  $\mathbf{k}$ -submodule of  $T(L)$  formed by the symmetric tensors (i.e., the tensors whose each graded components is invariant under all permutations of its tensorands). Decomposing  $T(L)^{\text{sym}}$  as the direct sum  $\bigoplus_{n \in \mathbb{N}} (L^{\otimes n})^{\text{sym}}$ , we realize that this diagonalizes  $\mathbf{t}'$ , because the operator  $\mathbf{t}'$  acts on the submodule  $(L^{\otimes n})^{\text{sym}} \cdot \text{Ker}(\mathbf{t}')$  as multiplication by  $n$ .

However,  $T(L) = T(L)^{\text{sym}} \cdot \text{Ker}(\mathbf{t}')$  does not hold when  $\mathbf{k}$  is merely torsionfree as an additive group and not a  $\mathbb{Q}$ -algebra. For instance, it fails for  $\mathbf{k} = \mathbb{Z}$ . Exactly how much of it can be salvaged in this generality remains a question.

The same questions can be asked about  $\mathbf{t}$ .

**TODO 9.3.** The operator  $\mathbf{t}'$  can be generalized to a sequence  $(\mathbf{t}'_1, \mathbf{t}'_2, \mathbf{t}'_3, \dots)$  of operators on  $T(L)$ , the  $N$ -th of which picks out  $N$  tensorands with increasing indices and moves them to the front. In other words,  $\mathbf{t}'_N : T(L) \rightarrow T(L)$  is the  $\mathbf{k}$ -linear map defined by

$$\begin{aligned} & \mathbf{t}'_N(u_1 \otimes u_2 \otimes \cdots \otimes u_k) \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_N \leq k} u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_N} \\ & \quad \otimes u_1 \otimes u_2 \otimes \cdots \otimes \widehat{u_{i_1}} \otimes \cdots \otimes \widehat{u_{i_2}} \otimes \cdots \otimes \widehat{\cdots} \otimes \cdots \otimes \widehat{u_{i_N}} \otimes \cdots \otimes u_k. \end{aligned}$$

These are somewhat similar to Schocker's operators in [Schock02] (but do not commute). I suspect that the intersection  $\text{Ker}(\mathbf{t}'_1) \cap \text{Ker}(\mathbf{t}'_2) \cap \cdots \cap \text{Ker}(\mathbf{t}'_N)$  is the algebra generated by  $L'_{N+1} + L'_{N+2} + L'_{N+3} + \cdots$  (when  $\mathbf{k}$  is a  $\mathbb{Q}$ -algebra and  $L$  is a free  $\mathbf{k}$ -module).

What I can prove is that the intersection  $K_N := \text{Ker}(\mathbf{t}'_1) \cap \text{Ker}(\mathbf{t}'_2) \cap \cdots \cap \text{Ker}(\mathbf{t}'_N)$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ , and that it contains  $L'_{N+1} + L'_{N+2} + L'_{N+3} + \cdots$ . Actually, this can all be rephrased as a question on Hopf algebras. Namely, for every  $p \in \mathbb{N}$ , let  $\pi_p$  be the projection  $T(L) \rightarrow L^{\otimes p}$  from the tensor algebra  $T(L)$  onto its  $p$ -th graded component. Recall that  $T(L)$  is a Hopf algebra, with comultiplication  $\Delta : T(L) \rightarrow T(L) \otimes T(L)$  given by

$$\begin{aligned} \Delta(u_1 u_2 \cdots u_k) \\ = \sum_{p=0}^k \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq k} \left( u_{i_1} u_{i_2} \cdots u_{i_p} \right) \otimes \left( u_1 u_2 \cdots \widehat{u_{i_1}} \cdots \widehat{u_{i_2}} \cdots \widehat{\cdots} \cdots \widehat{u_{i_p}} \cdots u_k \right), \end{aligned}$$

where we are using the  $\otimes$  symbol only for tensors inside  $T(L) \otimes T(L)$ , not for tensors inside  $T(L)$  (those latter tensors are simply written as products). For every  $N \in \mathbb{N}$ , the maps  $\mathbf{t}'_N : T(L) \rightarrow T(L)$  and  $(\pi_N \otimes \text{id}) \circ \Delta : T(L) \rightarrow L^{\otimes N} \otimes T(L)$  are therefore "equivalent" (i.e., they become equal if we canonically embed  $L^{\otimes N} \otimes T(L)$  into  $T(L)$  via the map  $a \otimes b \mapsto ab$ ). As the consequence, their kernels are equal. In other words,

$$\text{Ker}(\mathbf{t}'_N) = \text{Ker}((\pi_N \otimes \text{id}) \circ \Delta)$$

for every  $N \in \mathbb{N}$ . Now,

$$\begin{aligned} K_N &= \text{Ker}(\mathbf{t}'_1) \cap \text{Ker}(\mathbf{t}'_2) \cap \cdots \cap \text{Ker}(\mathbf{t}'_N) \\ &= \bigcap_{p=1}^N \underbrace{\text{Ker}(\mathbf{t}'_p)}_{\substack{= \text{Ker}((\pi_p \otimes \text{id}) \circ \Delta) \\ \text{(as shown above)}}} = \bigcap_{p=1}^N \text{Ker}((\pi_p \otimes \text{id}) \circ \Delta) \\ &= \left\{ x \in T(L) \mid \Delta(x) \in 1 \otimes x + \sum_{p=N+1}^{\infty} L^{\otimes p} \otimes T(L) \right\}. \end{aligned}$$

Now, why is  $L'_{N+1} + L'_{N+2} + L'_{N+3} + \cdots \subseteq K_N$ ? It clearly suffices to show that  $L'_u \subseteq K_N$  for any  $u > N$ . To prove this, we recall that the elements  $x$  of  $L'_u$  are primitive elements of  $T(L)$  (this is the easiest part of the Dynkin-Specht-Wever theorem); they therefore satisfy

$$\begin{aligned} \Delta(x) &= 1 \otimes x + \underbrace{x \otimes 1}_{\substack{\in \sum_{p=N+1}^{\infty} L^{\otimes p} \otimes T(L) \\ \text{(since } u > N)}} \in 1 \otimes x + \sum_{p=N+1}^{\infty} L^{\otimes p} \otimes T(L), \quad (30) \end{aligned}$$

which means that they lie in  $K_N$ .

It is also easy to prove that  $K_N$  is a  $\mathbf{k}$ -subalgebra of  $T(L)$ ; this relies on the bialgebra axiom  $\Delta(x)\Delta(y) = \Delta(xy)$  in the  $\mathbf{k}$ -bialgebra  $T(L)$ .

Now (why) is  $K_N$  actually the  $\mathbf{k}$ -subalgebra of  $T(L)$  generated by  $L'_{N+1} + L'_{N+2} + L'_{N+3} + \dots$ ?

Ah, I see it, at least in the case when  $\mathbf{k}$  is a field of characteristic 0. WLOG assume that  $L$  is finite free. Consider the shuffle Hopf algebra  $\text{Sh}(L^*) = \bigoplus_{\ell \in \mathbb{N}} \text{Sh}_\ell(L^*)$  which is the graded dual of  $T(L)$ . This algebra  $\text{Sh}(L^*)$  is commutative. Now,

$$\begin{aligned} K_N &= \bigcap_{p=1}^N \underbrace{\text{Ker}((\pi_p \otimes \text{id}) \circ \Delta)}_{=(\text{Sh}_p(L^*) \cdot \text{Sh}(L^*))^\perp} = \bigcap_{p=1}^N (\text{Sh}_p(L^*) \cdot \text{Sh}(L^*))^\perp \\ &= \left( \sum_{p=1}^N \text{Sh}_p(L^*) \cdot \text{Sh}(L^*) \right)^\perp. \end{aligned}$$

Since  $\sum_{p=1}^N \text{Sh}_p(L^*) \cdot \text{Sh}(L^*)$  is an ideal of  $\text{Sh}(L^*)$ , this yields (by basic properties of coalgebras) that its orthogonal space  $K_N$  is a subcoalgebra of  $T(L)$ . Thus,  $K_N$  is both a subalgebra and a subcoalgebra of  $T(L)$ . Since  $K_N$  is also graded, this shows that  $K_N$  is a connected graded  $\mathbf{k}$ -Hopf algebra. By the Cartier-Milnor-Moore theorem, this yields that  $K_N$  is generated by its primitive elements. Now, its primitive elements belong to  $L^{\otimes(N+1)} + L^{\otimes(N+2)} + L^{\otimes(N+3)} + \dots$  (by the argument we made in (30), reversed) and thus belong to  $L'_{N+1} + L'_{N+2} + L'_{N+3} + \dots$  (due to the Dynkin-Specht-Wever theorem in  $T(L)$ ). So  $K_N$  is the  $\mathbf{k}$ -subalgebra of  $T(L)$  generated by  $L'_{N+1} + L'_{N+2} + L'_{N+3} + \dots$  when  $\mathbf{k}$  is a field of characteristic 0.

Can we get rid of the requirement that  $\mathbf{k}$  be a field? I hope so, but this would require us use other methods. (For example, can we use the coradical filtration of  $T(L)$ ?)

## References

- [Lundkv08] Christian Lundkvist, *Counterexamples regarding symmetric tensors and divided powers*, Journal of Pure and Applied Algebra 212 (2008) 2236–2249. See arXiv:math/0702733v2 for a preprint.
- [Pang15] C. Y. Amy Pang, *Card-Shuffling via Convolutions of Projections on Combinatorial Hopf Algebras*, arXiv:1503.08368v1.
- [ReSaWe14] Victor Reiner, Franco Saliola, Volkmar Welker, *Spectra of Symmetrized Shuffling Operators*, Memoirs of the American Mathematical

Society, 2014, Volume: 228. ISBN-10: 0-8218-9095-6.  
See arXiv:1102.2460v2 for a preprint.

- [sage] SageMath, the Sage Mathematics Software System (Version 7.4.beta2), The Sage Developers, 2016, <http://www.sagemath.org>.
- [Schock02] Manfred Schöcker, *Idempotents for derangement numbers*, Discrete Mathematics, Volume 269, Issues 1–3, 28 July 2003, 239–248.  
<http://www.sciencedirect.com/science/article/pii/S0012365X02007574>
- [Specht50] Wilhelm Specht, *Gesetze in Ringen. I*, Mathematische Zeitschrift (1950), volume 52, 557–589.
- [Grinbe10] Darij Grinberg, *Strange boundary-like map on tensor algebra: what is its kernel?*, MathOverflow question #29923.  
<http://mathoverflow.net/q/29923>